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The Open University
Mathematics/Science/Technology
An Inter-faculty Second Level Course
MST204 Mathematical Models and Methods.

mathematical models and methods

unit 24 Normal modes

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Unit 24
Normal modes

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1.7

Contents

Introduction

This unit is about vibrations and it continues from where *Unit 8* left off. Its aim is to extend your ability to model systems in which vibrations play an important role. It will use some of the mathematical techniques you have learnt since you last studied vibrations in *Unit 8*, such as systems of differential equations, matrices and eigenvalues.

Study guide

Each section of this unit builds on those that have gone before. You should therefore read the unit in the order in which it is written. The first four sections all contain new material, while the fifth section pulls this material together by means of problems which are intended to revise all the main points. You should do as many of these as you can (or have time for) without first looking at the solutions, but you should then compare your answers with those given at the end of the unit. Finally look at the solutions to any of the problems which you did not attempt, because you will probably learn something from them, too.

There is no tape associated with this unit but there is a television programme. You should, if possible, have read at least the first two sections before you watch it—but don't miss it even if you haven't got that far. The programme contains demonstration experiments which are designed to make it easier to follow the theory in the unit.

1 Modelling vibrating systems

1.1 The story so far

In *Units 7* and *8* you studied the behaviour, both 'free' and 'forced', of some systems which were modelled by perfect springs, perfect dashpots and particles. The assembly in Figure 1 is typical of the models discussed in those units. What you did, in terms of Figure 1, was to write down a differential equation to represent the system with or without an applied force and then to obtain a solution in the form of an equation for x in terms of time and in terms of the parameters of the model (assumed constant): the stiffness k, the mass m and the damping coefficient r. You met a number of different models of this sort but in all cases the equation of the motion derived from the model took the form of a linear second-order differential equation with constant coefficients. You may well find it useful to revise some of this work by looking back at *Units 7* and *8*. Exercise 1 is designed to help you recall some of the material in these units and to prepare you for later sections of this unit. If you are in any doubt about any part of this material you should consult *Units 7* and *8*.

The behaviour of a vibrating system is **forced** if it is brought about by an external forcing agency (see *Unit 8*, Section 2); otherwise the behaviour is said to be **free**.

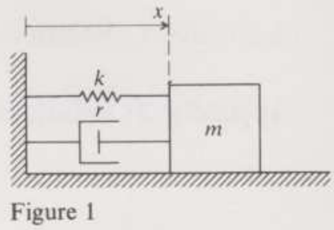


Figure 1

Exercise 1 (*Revision*)

Suppose that in Figure 1, $m = 0.1$, $k = 2 \times 10^6$ and $r = 0$. Write down the equation of motion for free vibrations and work out the angular frequency of the vibrations.

[*Solution on p. 31*]

Unless otherwise stated, all quantities in this unit are expressed in terms of SI units. Thus the mass of the particle is 0.1 kilograms and the stiffness of the spring is 2×10^6 newtons per metre.

In *Unit 7* the general solution of the equation of motion for free vibrations of the model in Exercise 1 was shown to be $x = x_0 + C\sin\omega t + D\cos\omega t$, where x_0, C, D and ω are constants. (ω is the 'natural' angular frequency which you worked out in Exercise 1. The constants C and D are arbitrary.) This is not the only form for the general solution. By choosing A and ϕ such that $C = A\cos\phi$ and $D = A\sin\phi$ we obtain

$$\begin{aligned} C\sin\omega t + D\cos\omega t &= A\cos\phi\sin\omega t + A\sin\phi\cos\omega t \\ &= A\sin(\omega t + \phi), \end{aligned}$$

so the general solution can be written in the alternative form

$$x = x_0 + A\sin(\omega t + \phi). \qquad (1)$$

It is usual to choose A to be positive and $-\pi < \phi \leqslant \pi$, in which case

$$A = \sqrt{C^2 + D^2}$$

and

$$\phi = \begin{cases} \arccos\left(\dfrac{C}{A}\right) & \text{if } D \geqslant 0 \\ -\arccos\left(\dfrac{C}{A}\right) & \text{if } D < 0. \end{cases}$$

This is a more convenient formulation for our present purpose and we shall use it throughout this unit.

We can consider Solution (1) to be made up of two parts. The sinusoidal term $A\sin(\omega t + \phi)$ represents the vibration. The constant x_0 gives the static position which the particle occupies when it is not vibrating (mathematically the static solution corresponds to setting the arbitrary constant A to zero). The value of x_0 depends on our choice of fixed origin from which x is measured. If x is measured from the anchored end of the spring (as in Figure 1), x_0 is equal to the spring's natural length. In this unit however we shall usually choose the origin to be at the static position so that $x_0 = 0$, thereby obtaining a simpler expression for the general solution. I shall have more to say about this in Section 2.

1.2 Lumped-parameter models

The perfect springs, perfect dashpots and particles which make up the models in *Units 7* and *8* are *idealized* components. Each component represents a particular physical property. For example, perfect springs are assumed only to have resilience (i.e. springiness): they deform under load in accordance with Hooke's law and return to their original length when the load is removed. They are assumed to have no mass and no damping properties. Perfect dashpots provide linear damping but no mass or resilience and particles have mass but no resilience or damping properties. These are not, of course, completely accurate descriptions of real components: a real spring must have some mass and its material will exhibit at least some damping properties; no real body can behave exactly like a particle; it is bound to deform to some extent under the action of the forces to which it is subject—and so on. Nevertheless, models consisting of idealized components can be used to describe a wide variety of real systems quite adequately.

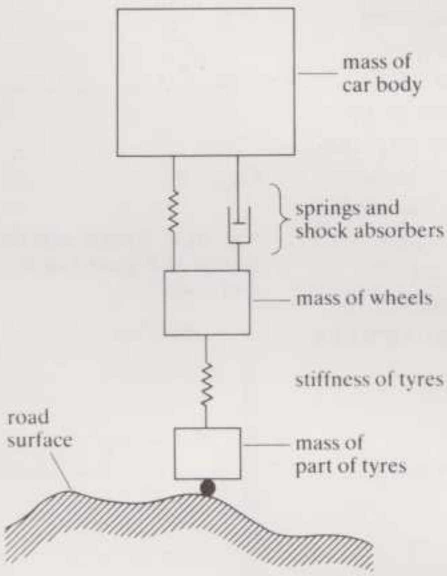


Figure 2

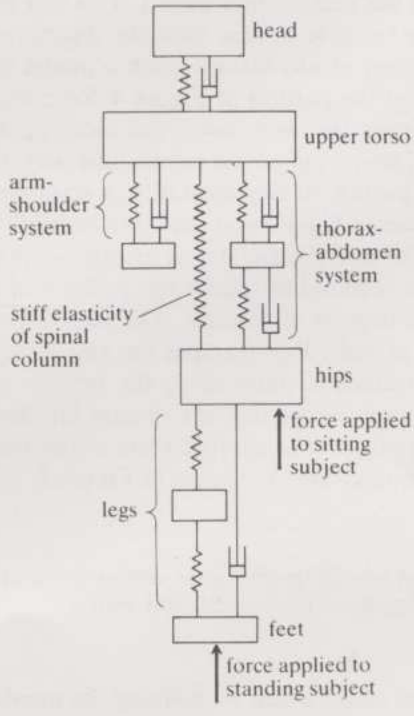


Figure 3

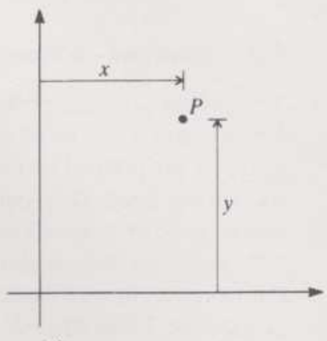
As an example of how this can be done consider the vertical vibration of a motor car as it moves over an uneven road surface. Figure 2 shows a deliberately simplified model for estimating the magnitude of these vibrations for a particular road profile and car speed. The model consists of idealized components each of which represents a physical property of the car. To keep the model as simple as possible we confine our attention to those properties of the car that can be expected to have a *significant* influence on its vertical motion. We can, for instance, expect the car's suspension to play an important role in contributing both resilience and damping to the car's vertical motion and this is represented in the model by a perfect spring and a perfect dashpot. Similarly, the resilience of the

tyres is important and is represented in the model by a second perfect spring. The particles in the model are used to represent the masses of the car body, the wheels, and the tyres. Since the model is only intended to yield information about the vertical motion of the car the particles are considered to move along one vertical straight line.

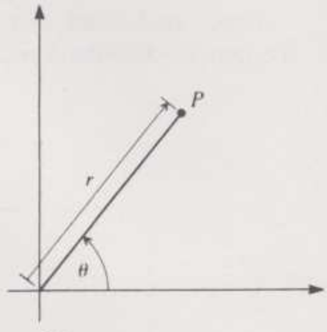
It is important to realize that in constructing such models we do not attempt to describe the detailed structure of the device we are modelling. Rather we try to identify its important properties (mass, resilience, damping) and 'lump' them into appropriate idealized components. This kind of model is therefore known as a **lumped-parameter model**.

The applications of lumped-parameter models are many and varied. I shall mention just one more example that was built up in the course of investigations into the effect of vibration on the human body. It was found that different combinations of amplitude and frequency have different effects, ranging from painful to imperceptible. The model in Figure 3 was found useful in predicting the reaction of a standing or sitting human being to vibrations at low frequencies. Once again it is valid only for up and down vibrations so that particles in the model are free to move only along vertical straight lines.

The accuracy of the models in Figures 2 and 3 could well have been increased by incorporating more elements. Unfortunately this would also increase the complexity of the resulting equations of motion. In general, when designing lumped-parameter models a balance must be struck between the accuracy of the model and the complexity of the mathematics.



(a)



(b)

Figure 4

1.3 Degrees of freedom

Consider again the motion of the particle in Figure 1. This can be represented by the variation with time of one variable, x. One variable, therefore, is enough to specify the position of the particle at any instant: such a model is said to have one degree of freedom. Now look at the particle in Figure 4. Suppose that it is free to move in the plane of the diagram: that is to say it can move up and down and sideways but it cannot move into or out of the paper. One way, therefore, of specifying the position of the particle at any instant is to state the values of the displacements x and y as shown in Figure 4(a). Another possibility would be to state the length r of the line OP and the angle θ as in Figure 4(b). Either way, two numbers (or 'coordinates') are required to specify the position of the particle and so we say the particle has two degrees of freedom. The same terminology is used when there is more than one particle. For example, the two spring-connected particles in Figure 5 are constrained to move along the straight line which connects their centres. So the only things that can change are the distances of the two particles from some fixed origin. The configuration of the system can therefore be specified by the two distances x_1 and x_2 shown in Figure 5.

You have already met the system in Figure 5 in Section 4 of *Unit 7*.

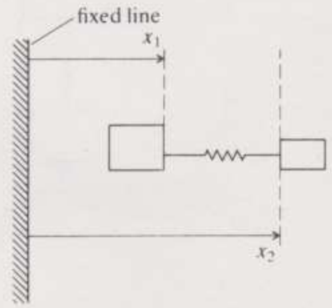


Figure 5

Exercise 2

Make a sketch to show a different way of specifying the configuration of the two-particle system in Figure 5. How many degrees of freedom has this system?

[*Solution on p. 31*]

What I have said so far can be summarized by defining the **number of degrees of freedom** of a system to be the smallest number of coordinates which are required to specify the configuration of the system at any instant. The importance of the number of degrees of freedom is that it is equal to the number of equations of motion.

If a lumped-parameter model is such that each particle is free to move in just one direction, the number of degrees of freedom is equal to the number of particles. This is because the configuration of the model can be specified by giving the displacement of each particle along its direction of motion. For example, the lumped-parameter model in Figure 2 (modelling the vertical vibrations of a car) has three degrees of freedom because it is made up of three particles that are free to move vertically.

Exercise 3

How many degrees of freedom has the model in Figure 3?

[*Solution on p. 31*]

Exercise 4

State the number of degrees of freedom for each of the lumped-parameter models shown in Figure 6 below. (The particles are constrained to move only along the lines connecting the particles.)

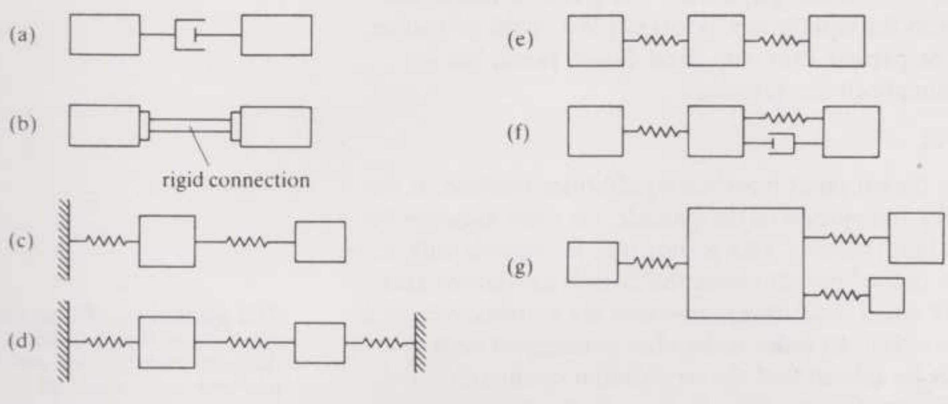


Figure 6

[*Solution on p. 31*]

Exercise 5

A particle is free to move in space. How many degrees of freedom does it have?

[*Solution on p. 31*]

The rest of this unit is an introduction to the mechanics of lumped-parameter models with more than one degree of freedom. We shall confine our attention to models of systems in which damping does not play an important role. We begin, in the next section, by considering a model with two degrees of freedom.

Summary of Section 1

This unit models real systems by combinations of idealized components each of which has only one of the relevant properties, i.e. mass, stiffness or damping. Such combinations are called **lumped-parameter models**. This unit deals with the analysis of undamped lumped-parameter models.

The least number of coordinates that is required to define the configuration of such a model is called the **number of degrees of freedom**.

2 Free undamped vibrations with two degrees of freedom

The vibration of lumped-parameter models with more than one degree of freedom gives rise to phenomena which are not observed in a one-degree-of-freedom model. This section tackles the analysis of the simplest model that will unequivocally show up these phenomena. This is a model with two degrees of freedom (it is shown in Figure 1) and it has all the important features which are found in more complicated undamped lumped-parameter models.

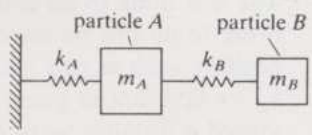


Figure 1

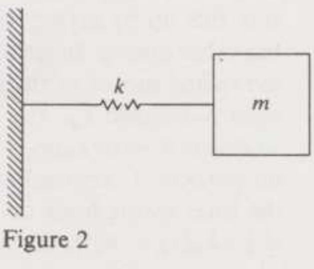


Figure 2

2.1 Setting up the equations of motion

Before we can set up the equations of motion for the model in Figure 1 we must decide on the coordinates we are going to use to measure the positions of the two particles. To this end we first consider the simpler model in Figure 2 which only

has one degree of freedom. If the particle in Figure 2 is displaced from its equilibrium position and released, it will move with simple harmonic motion about the equilibrium position. As I remarked in Section 1, we can obtain a simple expression for the motion by measuring the displacement x from the equilibrium position. The expression we obtain is then

By **equilibrium** position I mean the **static** position (i.e. the position at which there is no net force acting on the particle).

$$x = A\sin(\omega t + \phi)$$

where ω is the natural undamped angular frequency and A and ϕ are constants with A positive and $-\pi < \phi \leqslant \pi$. ($A$ is the amplitude — it gives the maximum displacement of the particle from the equilibrium position.) We could, of course, measure the displacement of the particle from any fixed datum point, but we would then obtain the more complicated expression

$$x = x_0 + A\sin(\omega t + \phi)$$

where x_0 is the distance of the datum point from the equilibrium position. If we want the simplest expression for the motion of the particle, we must measure the displacement from the equilibrium position. This is also true for models with more than one degree of freedom — indeed the simplification is even greater for such models, and that is why in this unit I shall always measure the displacement of a particle from its equilibrium position. In order to be able to interpret such measurements you will need to be able to find the equilibrium configuration of lumped-parameter models. Here are two exercises to give you practice at this.

This method of defining the positions of the particles is different from the one used in previous units where all positions were referred to a single origin. The convention used in this unit is much more convenient for our present purposes.

Exercise 1

Each of the two springs of the lumped-parameter model in Figure 3 has a natural length of 0.6 m and a stiffness of $75\,\mathrm{N\,m^{-1}}$. How far are the particles from the fixed datum line when both particles are in equilibrium? (Remember that particles are considered to have negligible size.)

[*Solution on p. 31*]

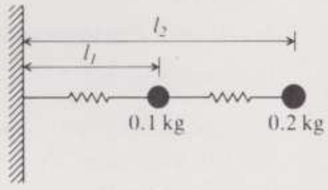


Figure 3

Exercise 2

The model in Figure 3 is suspended vertically from a fixed support, so that the 0.2 kg particle is below the 0.1 kg particle and both are free to move in a vertical straight line. How far are the particles below the support when both particles are in equilibrium?

[*Solution on p. 31*]

We shall now return to the model in Figure 1. This has been redrawn in Figure 4. Two configurations are shown: Figure 4(a) shows the equilibrium configuration in which both springs are unloaded (so that they both have their natural length); Figure 4(b) shows an arbitrary configuration such as might occur instantaneously during vibrations. The displacements of the particles A and B from their equilibrium positions will be called respectively, x_A and x_B. The direction indicated by the single arrowheads represents the direction of displacement, velocity, acceleration and force which I propose to consider as positive. The choice of positive direction is, as usual, arbitrary, but must always be made clear and must be adhered to consistently in all subsequent work.

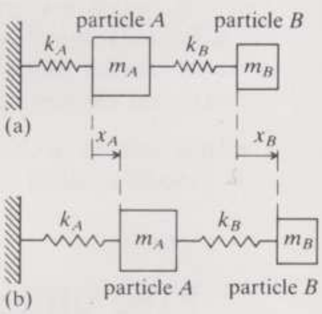


Figure 4

The first thing to do is to write down the equation of motion for each particle. In order to do this we must know the forces due to the springs, in any arbitrary position such as that shown in Figure 4(b). Looking first at the spring of stiffness k_A we find that in going from Figure 4(a) to Figure 4(b) its right-hand end has moved a distance x_A to the right and its left-hand end has remained fixed. The spring has therefore been stretched by an amount x_A and exerts a force of magnitude $k_A x_A$ acting to the left on particle A. Using our sign convention we can sum this up by saying that the spring exerts a force $-k_A x_A$ on particle A. Now for the other spring. In going from Figure 4(a) to Figure 4(b) the right-hand end of the spring moves to the right a distance x_B and its left-hand end moves to the right a distance x_A. The spring is therefore stretched by an amount $(x_B - x_A)$ and so exerts a force $k_B(x_B - x_A)$ which acts to the left on particle B and to the right on particle A. Remembering that positive forces act to the right we can say that the total spring force on particle A is $\{k_B(x_B - x_A) - k_A x_A\}$ and that on particle B is $\{-k_B(x_B - x_A)\}$. Since in our model the spring forces are the only forces acting

on the particles in the direction of motion we can now write down the equations of motion of both particles by invoking Newton's second law:

$$k_B(x_B - x_A) - k_A x_A = m_A \ddot{x}_A \tag{1}$$

$$-k_B(x_B - x_A) = m_B \ddot{x}_B. \tag{2}$$

After dividing the first equation by m_A, the second by m_B and then rearranging, we can rewrite (1) and (2) in the matrix form

$$\begin{bmatrix} -\dfrac{(k_A + k_B)}{m_A} & \dfrac{k_B}{m_A} \\ \dfrac{k_B}{m_B} & -\dfrac{k_B}{m_B} \end{bmatrix} \begin{bmatrix} x_A \\ x_B \end{bmatrix} = \begin{bmatrix} \ddot{x}_A \\ \ddot{x}_B \end{bmatrix}. \tag{3}$$

Thus (3) is a system of differential equations which describes the behaviour of the model in Figure 4.

The next task is to find the solutions of (3). That is, find expressions for x_A and x_B as functions of time. There are, of course, many different solutions of (3) but in the next subsection we shall show that it is possible to find some particularly simple solutions, called normal modes, with the property that both particles execute simple harmonic motion at the same angular frequency. The importance of these normal mode solutions will become apparent at the end of this section where we show that *any* solution of (3) can be expressed as a sum of normal modes.

2.2 Normal modes

One way of solving the equation of motion (3) would be to turn immediately to the method described in the unit on systems of differential equations. Using this method, one can obtain the general solution of the equations of motion. However, before doing this I shall try to gain a more intuitive feeling for the behaviour of the system by considering some special solutions which can be obtained without reference to the simultaneous differential equations unit. To find them, let us recall that the typical motion of an undamped vibrating system with only *one* degree of freedom is simple harmonic motion, and ask ourselves whether a similar motion is possible with two degrees of freedom. More specifically, we ask ourselves the question: is it possible for the two-degrees-of-freedom system in Figure 1 to move in such a way that both particles execute simple harmonic motion with the same angular frequency? To answer this question let us see whether it is possible to satisfy Equation (3) with sinusoidal functions:

$$x_A(t) = A \sin(\omega t + \phi_A)$$

$$x_B(t) = B \sin(\omega t + \phi_B)$$

in which A, ϕ_A, B, ϕ_B and ω are constants to be determined. Without loss of generality we shall impose the constraints that both A and B (the amplitudes) and ω (the angular frequency) are positive and that both ϕ_A and ϕ_B lie between $-\pi$ and π.

Using the above expressions for x_A and x_B, it follows that $\dot{x}_A = A\omega \cos(\omega t + \phi_A)$ and $\dot{x}_B = B\omega \cos(\omega t + \phi_B)$ and, differentiating a second time, that $\ddot{x}_A = -A\omega^2 \sin(\omega t + \phi_A)$ and $\ddot{x}_B = -B\omega^2 \sin(\omega t + \phi_B)$. We can rewrite the last two expressions more simply as follows:

$$\ddot{x}_A = -\omega^2 x_A \tag{4}$$

$$\ddot{x}_B = -\omega^2 x_B. \tag{5}$$

Equations (4) and (5) are, of course, merely alternative statements of the fact that we are assuming that the two particles move with simple harmonic motion about their equilibrium positions, the frequency being the same for both particles. In matrix form Equations (4) and (5) can be written

$$\begin{bmatrix} \ddot{x}_A \\ \ddot{x}_B \end{bmatrix} = -\omega^2 \begin{bmatrix} x_A \\ x_B \end{bmatrix}.$$

Substituting this expression into Equation (3) gives

$$\begin{bmatrix} -\dfrac{(k_A + k_B)}{m_A} & \dfrac{k_B}{m_A} \\ \dfrac{k_B}{m_B} & -\dfrac{k_B}{m_B} \end{bmatrix} \begin{bmatrix} x_A \\ x_B \end{bmatrix} = -\omega^2 \begin{bmatrix} x_A \\ x_B \end{bmatrix}. \tag{6}$$

From the unit on eigenvalues we know that Equation (6) has a solution provided $-\omega^2$ is an eigenvalue of the matrix on the left-hand side. So, for each eigenvalue $-\omega^2$ there are solutions of Equation (3) in which both particles are oscillating with the same angular frequency ω. Such solutions are called **normal modes** and the angular frequency of the motion is called the **normal mode angular frequency**.

Finding the normal mode angular frequencies

To avoid an excessive amount of algebra I shall illustrate how to calculate the normal mode frequencies by working through an example with particular values for m_A, m_B, k_A and k_B.

Example 1

Suppose that the parameters for the model shown in Figure 1 are $m_A = 6$, $m_B = 4$, $k_A = 20$ and $k_B = 10$. Calculate the normal mode angular frequencies of the model.

Solution

Substituting the values of m_A, m_B, k_A and k_B into Equation (6) gives

$$\begin{bmatrix} -5 & 1.67 \\ 2.5 & -2.5 \end{bmatrix} \begin{bmatrix} x_A \\ x_B \end{bmatrix} = -\omega^2 \begin{bmatrix} x_A \\ x_B \end{bmatrix}.$$

All calculations in this unit are worked to the full accuracy of a calculator, but are recorded here to two decimal places for convenience.

Since the matrix is 2×2 its eigenvalues are easily found by solving the characteristic equation. Here I shall call the eigenvalues $-\omega^2$, rather than λ as in the unit on eigenvalues, and so the characteristic equation takes the form

$$\begin{vmatrix} -5 + \omega^2 & 1.67 \\ 2.5 & -2.5 + \omega^2 \end{vmatrix} = 0.$$

Since it is the values of the angular frequencies ω that interest us, rather than the eigenvalues $-\omega^2$, we can proceed as follows: we expand the determinant to give the equation

$$(\omega^2)^2 - 7.5(\omega^2) + 8.33 = 0,$$

and then solve for ω^2 by the usual method for quadratics. This gives

$$\omega^2 = \frac{7.5 \pm \sqrt{7.5^2 - 4 \times 8.33}}{2}$$

$$= 1.36 \text{ or } 6.14,$$

and so the possible values for the angular frequency ω are

$$\omega = \sqrt{1.36} = 1.16 \quad \text{and} \quad \omega = \sqrt{6.14} = 2.48.$$

There are therefore two possible frequencies at which both particles of the two-degrees-of-freedom model can move with simple harmonic motion. More briefly, the model has two normal mode angular frequencies. I shall make it a rule throughout this unit to call the lowest value normal mode angular frequency ω_1 and to number the other values in order of magnitude. The normal mode with angular frequency ω_1 is called the **first** normal mode, the one with angular frequency ω_2 is called the **second** normal mode — and so on. In this case we have

$$\omega_1 \simeq 1.16\,(\text{rad s}^{-1}) \quad \text{and} \quad \omega_2 \simeq 2.48\,(\text{rad s}^{-1}).$$

Now try some similar calculations for yourself:

Exercise 3

Assuming that in the model shown in Figure 4 the stiffnesses are $k_A = k_B = k$ and the masses are $m_A = m_B = m$, calculate the two normal mode angular frequencies. Do the calculations as accurately as you can but give the numbers in the final result accurate to two places of decimals.

[*Solution on p. 31*]

Exercise 4

Suppose that the parameters for the model shown in Figure 1 are $m_A = 1.67 \times 10^3$, $m_B = 167$, $k_A = 12 \times 10^6$ and $k_B = 5.9 \times 10^6$. Calculate the two normal mode angular frequencies. Do the calculations as accurately as you can but give the final results to the nearest integer.

[*Solution on p. 31*]

To sum up what we have done so far: we began by investigating the conditions under which the two particles of the model in Figure 4 could both move with simple harmonic motion at the same angular frequency. We did this by looking for special solutions of the equation of motion (3), having the form

$$\begin{bmatrix} x_A(t) \\ x_B(t) \end{bmatrix} = \begin{bmatrix} A\sin(\omega t + \phi_A) \\ B\sin(\omega t + \phi_B) \end{bmatrix} \tag{7}$$

and found that there are *two* angular frequencies ($\omega = \omega_1$ and $\omega = \omega_2$) for which such solutions exist. The motions described by these special solutions are called normal modes and the angular frequencies ω_1 and ω_2 are called the normal mode angular frequencies.

Although we have seen how the normal mode angular frequencies ω_1 and ω_2 are found, as yet we have no information concerning the amplitudes and phases of the normal modes. We turn our attention to this now.

Normal mode displacement ratios

To a certain extent the amplitudes and phases of the normal mode oscillations are arbitrary. They depend on the starting positions and velocities of the particles and cannot be determined without additional information. What we can do however is find a relationship between the phases ϕ_A and ϕ_B, and between the amplitudes A and B, when the particles are executing normal mode motion. To do this we go back to the conditions for normal mode motion given by the eigenvector equation (6):

$$\begin{bmatrix} -\dfrac{(k_A + k_B)}{m_A} & \dfrac{k_B}{m_A} \\ \dfrac{k_B}{m_B} & -\dfrac{k_B}{m_B} \end{bmatrix} \begin{bmatrix} x_A(t) \\ x_B(t) \end{bmatrix} = -\omega^2 \begin{bmatrix} x_A(t) \\ x_B(t) \end{bmatrix}. \tag{6}$$

We have seen that normal mode motion occurs when $-\omega^2$ is an eigenvalue of the matrix on the left-hand side. But so far we have not considered the eigenvectors. For normal mode motion the particles must move in such a way that $[x_A(t) \;\; x_B(t)]^T$ is always equal to an eigenvector corresponding to the eigenvalue $-\omega^2$. This means that $x_A(t)$ and $x_B(t)$ must satisfy the equations

$$\left(\omega^2 - \frac{(k_A + k_B)}{m_A}\right) x_A(t) + \frac{k_B}{m_A} x_B(t) = 0 \tag{7}$$

$$\frac{k_B}{m_B} x_A(t) + \left(\omega^2 - \frac{k_B}{m_B}\right) x_B(t) = 0. \tag{8}$$

From either of these equations we obtain

$$x_B(t) = R x_A(t) \tag{9}$$

where R is a constant given by

$$R = \frac{k_B + k_A - m_A\omega^2}{k_B} = \frac{k_B}{k_B - m_B\omega^2}.$$

The first expression for R comes from Equation (7); the second from Equation (8). Provided ω is a normal mode angular frequency the two expressions are equal.

From (9) we have

$$\begin{bmatrix} x_A(t) \\ x_B(t) \end{bmatrix} = \begin{bmatrix} x_A(t) \\ R x_A(t) \end{bmatrix} = x_A(t) \begin{bmatrix} 1 \\ R \end{bmatrix}.$$

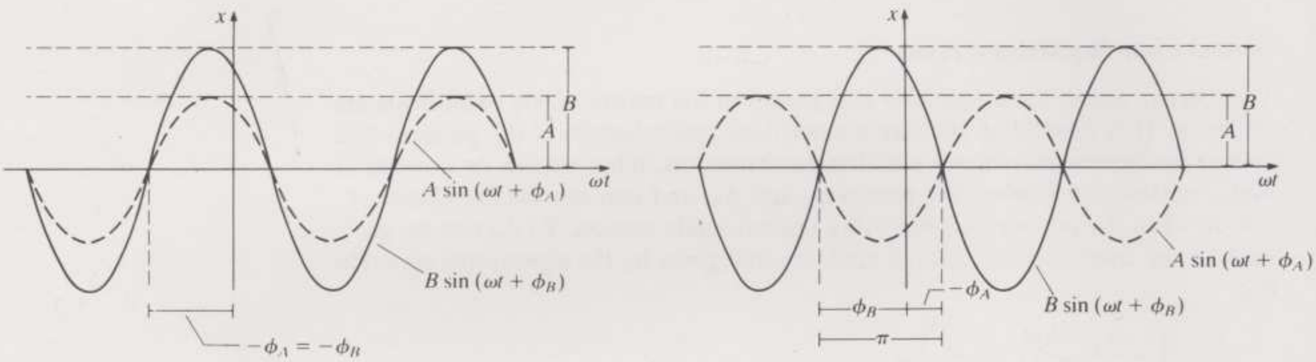
So for normal mode motion $[x_A(t) \;\; x_B(t)]^T$ is equal to the eigenvector $[1 \; R]^T$ multiplied by the sinusoidal function $x_A(t) = A\sin(\omega t + \phi_A)$, or in terms of the component functions $x_A(t)$ and $x_B(t)$:

$$x_A(t) = A\sin(\omega t + \phi_A)$$

$$x_B(t) = RA\sin(\omega t + \phi_A).$$

Since R is a constant, Equation (9) tells us that the two displacements $x_A(t)$ and $x_B(t)$ are proportional, and that the ratio $x_A(t)/x_B(t)$ (for $x_B(t)$ non-zero) is equal to R for all values of t. For this reason R is called the **displacement ratio** of the normal mode.

Because $x_A(t)$ and $x_B(t)$ are proportional, the two particles pass through their static positions at the same time. There are two possible types of motion, depending on the sign of the displacement ratio R. If R is positive, then (9) shows that x_A and x_B always have the same sign so that the two particles are always both on the same side of their static positions. In this case the two particles are said to be **in phase** (see Figure 5(a)). On the other hand if R is negative, then x_A and x_B always have opposite signs, so that both particles are always on opposite sides of their static positions. In this case the two particles are said to be **phase-opposed** (see Figure 5(b)).



(a) In-phase motion

(b) Phase-opposed motion

Figure 5

By looking carefully at the graphs in Figure 5 we can establish a relationship between the amplitudes A and B and between the phases ϕ_A and ϕ_B. We start by considering amplitudes. If we stretch the graph of x_A in each of Figures 5(a) and 5(b) by the factor $|R|$ (that is, by the magnitude of R) then it will have the same amplitude as the graph of x_B. This observation enables us to write down the following relationship between the amplitudes:

$$B = |R|A,$$

or

$$|R| = \frac{B}{A}.$$

So the magnitude of R gives the ratio of the amplitudes.

Let us now turn our attention to the phases. It is convenient to consider the cases of in-phase motion and phase-opposed motion separately. If the particles are in phase as in Figure 5(a) then stretching the graph of x_A vertically by $|R|$ will actually make it coincide with the graph of x_B. For this to happen both oscillations must have the same phase. So in this case we have

$$\phi_A = \phi_B.$$

If, however, the particles are phase-opposed then, in addition to stretching the graph of x_A vertically by $|R|$, we will have to shift it sideways a distance π to make

it coincide with the graph of x_B. So we conclude that the phase of x_B must differ from the phase of x_A by π. That is,

$$|\phi_B - \phi_A| = \pi.$$

In the expression $x_B(t) = RA\sin(\omega t + \phi_A)$ given above, the phase difference is automatically taken care of by the sign of R. If, however, we want an expression for $x_B(t)$ explicitly in terms of amplitude and phase we would have to write

$$x_B(t) = |R|A\sin(\omega t + \phi_B)$$

where $\phi_B = \phi_A$ if $R > 0$ or $|\phi_B - \phi_A| = \pi$ (with $-\pi < \phi_B \leq \pi$) if $R < 0$.

To summarize, each normal mode has a displacement ratio that specifies the relationship between the two particles: the *magnitude* of the displacement ratio gives the ratio of the amplitudes of the oscillations; the *sign* of the displacement ratio determines whether the oscillations are in phase or phase-opposed. If we denote the displacement ratio for the first normal mode by R_1 and the displacement ratio for the second normal mode by R_2 then the motion of the particles in the first normal mode is given by

$$x_A = A_1\sin(\omega_1 t + \phi_1), \qquad x_B = A_1R_1\sin(\omega_1 t + \phi_1)$$

where A_1 and ϕ_1 are constants (ϕ_1 was previously called ϕ_A). The motion in the second normal mode is given by

$$x_A = A_2\sin(\omega_2 t + \phi_2), \qquad x_B = A_2R_2\sin(\omega_2 t + \phi_2).$$

Finding the normal mode displacement ratio

As an illustration we shall work out the normal mode displacement ratios for the model in Example 1.

Example 2

In Example 1 we considered the model shown in Figure 1 with $m_A = 6$, $m_B = 4$, $k_A = 20$ and $k_B = 10$.

Calculate the normal mode displacement ratios for this model.

Solution

In Example 1 we considered the eigenvalue problem

$$\begin{bmatrix} -5 & 1.67 \\ 2.5 & -2.5 \end{bmatrix}\begin{bmatrix} x_A \\ x_B \end{bmatrix} = -\omega^2\begin{bmatrix} x_A \\ x_B \end{bmatrix}$$

and found two normal mode angular frequencies ω_1 and ω_2, where $\omega_1^2 = 1.36$ and $\omega_2^2 = 6.14$. To find the corresponding normal mode displacement ratios R_1 and R_2 we look for eigenvectors of the form $[1\ \ R_1]^T$ and $[1\ \ R_2]^T$. We do this in the usual way by substituting the appropriate values of ω^2 into the equation

$$\begin{bmatrix} -5+\omega^2 & 1.67 \\ 2.5 & -2.5+\omega^2 \end{bmatrix}\begin{bmatrix} x_A \\ x_B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad (9)$$

and solving for x_A and x_B.

For the first normal mode we put $\omega^2 = 1.36$ into (9) to obtain the simultaneous equations

$$-3.64\,x_A + 1.67\,x_B = 0$$
$$2.5\,x_A - 1.14\,x_B = 0.$$

Both these equations give

$$x_B = 2.19\,x_A \qquad \text{(to two places of decimals)}$$

and so putting $x_A = 1$ we obtain the required eigenvector $[1\ \ 2.19]^T$. It follows that the normal mode displacement ratio corresponding to the normal mode angular frequency ω_1 is $R_1 \simeq 2.19$.

For the second normal mode, we put $\omega^2 = 6.14$ into (9) to obtain the simultaneous equations

$$1.14\,x_A + 1.67\,x_B = 0$$

$$2.5\,x_A + 3.64\,x_B = 0.$$

Both these equations give

$$x_B = -0.69x_A \qquad \text{(to two places of decimals)}$$

and so putting $x_A = 1$ we obtain the required eigenvector $[1 \;\; -0.69]^T$. It follows that the second normal mode displacement ratio is $R_2 = -0.69$. The negative sign means that the motions of the two particles are phase-opposed so that when one particle has reached the extreme point of its travel to the right of its static position, the other particle will be at the extreme point in its travel to the left of its static position. It also means that during this motion the velocities of the two particles will always be in opposite directions (except, of course, when the velocities are instantaneously zero at the extreme ends of the travel of each particle).

The general undamped lumped-parameter model with two degrees of freedom

In this subsection I have dealt almost exclusively with the model shown in Figure 1 (page 7). However, all that I have said can be applied equally well to any undamped lumped-parameter model with two degrees of freedom. This is because the equations of motion for such models can always be written in the matrix form

$$\mathbf{Hx} = \ddot{\mathbf{x}}, \tag{10}$$

where $\mathbf{H}$ is a 2×2 matrix, and this is a straightforward generalization of the equation of motion (3) for the model in Figure 1. Indeed, we can obtain (3) from (10) by putting

$$\mathbf{H} = \begin{bmatrix} -\dfrac{(k_A + k_B)}{m_A} & \dfrac{k_B}{m_A} \\ \dfrac{k_B}{m_B} & -\dfrac{k_B}{m_B} \end{bmatrix}.$$

The techniques developed so far in this subsection have been based on Equation (3) but all these techniques can be applied to the more general Equation (10), as described in the following procedure.

Procedure 2.2: To find the normal modes of an undamped lumped-parameter model with two degrees of freedom

1. Write down the equations of motion for the system and express them in the matrix form

$$\mathbf{Hx} = \ddot{\mathbf{x}}.$$

2. Write down the characteristic equation of the matrix $\mathbf{H}$ in terms of $-\omega^2$ and solve for ω to obtain the two normal mode angular frequencies ω_1 and ω_2.

3. Find corresponding eigenvectors in the form $[1 \;\; R_1]^T$ and $[1 \;\; R_2]^T$ to obtain the two normal mode displacement ratios R_1 and R_2.

4. The motion of the particles in the first normal mode is given by

$$x_A = A_1 \sin(\omega_1 t + \phi_1) \qquad \text{and} \qquad x_B = R_1 A_1 \sin(\omega_1 t + \phi_1)$$

and the motion of the particles in the second normal mode is given by

$$x_A = A_2 \sin(\omega_2 t + \phi_2) \qquad \text{and} \qquad x_B = R_2 A_2 \sin(\omega_2 t + \phi_2).$$

You may wonder what happens if the matrix **H** in this procedure has positive eigenvalues, for it would then be impossible to find real values for the normal mode angular frequencies. It turns out, however, that the matrices **H** resulting from lumped-parameter models can never have positive eigenvalues. The only complication that can arise is for one of the eigenvalues to be zero. We shall deal with such cases in Section 4.

Exercise 5

Using the data and results of Exercise 4 determine the corresponding values of the normal mode displacement ratios.

[*Solution on p. 32*]

Exercise 6

Calculate the normal mode displacement ratios for the model in Exercise 3.

[*Solution on p. 32*]

2.3 The general motion

So far, all that we have done has concerned the normal mode motion of simple idealized systems like the one shown in Figure 1 (on page 7). Normal mode motion means that both particles move with simple harmonic motion at the same angular frequency, though generally with different amplitudes. But suppose you actually made up a system like the one in Figure 1, displaced each of the two particles by an arbitrary amount from its equilibrium position and then released both particles: you would probably find that neither particle moved with simple harmonic motion. Indeed the motions would probably look quite irregular and complicated. The television programme associated with this unit includes a demonstration of this. So the question arises: Is there any connection between the kind of motion I have just described and the normal modes? The answer is 'yes'; and, what is more, if we know the normal mode frequencies and displacement ratios, then we can predict the motions of the particles which result from any arbitrary set of initial conditions.

To justify this statement we go back to the equations of motion. From the last subsection we know that the equations of motion for an undamped lumped-parameter system with two degrees of freedom can be written as a system of differential equations of the form

$$\mathbf{Hx} = \ddot{\mathbf{x}}. \tag{10}$$

This type of system was discussed in the unit on the systems of differential equations, where (for negative eigenvalues) the general solution was shown to be

See *Unit 22*, Section 3.

$$\mathbf{x} = \sum_{r=1}^{n} \mathbf{a}_r(C_r\cos(\sqrt{-\lambda_r}\,t) + D_r\sin(\sqrt{-\lambda_r}\,t)).$$

In this solution n is the number of equations in the system, λ_r is the rth eigenvalue of **H**, $\mathbf{a}_r$ is the rth eigenvector of **H** and the C_r's and D_r's are arbitrary constants. We can now rewrite this general solution so as to conform with the notation developed during our discussion of normal modes in the last subsection.

In the last subsection we confined our attention to models with two degrees of freedom. For such models $n = 2$. Also, we used the notation $-\omega_r^2$ for the eigenvalue λ_r and so we can replace $\sqrt{-\lambda_r}$ by ω_r. Furthermore the eigenvector corresponding to the eigenvalue $-\omega_r^2$ was chosen to be of the form $[1\ \ R_r]^T$ where R_r is the rth normal mode displacement ratio. Thus in the notation of this unit the general solution of (10) is

$$\mathbf{x} = \begin{bmatrix} 1 \\ R_1 \end{bmatrix}(C_1\cos\omega_1 t + D_1\sin\omega_1 t) + \begin{bmatrix} 1 \\ R_2 \end{bmatrix}(C_2\cos\omega_2 t + D_2\sin\omega_2 t)$$

or in terms of x_A and x_B:

$$\left.\begin{aligned} x_A(t) &= (C_1\cos\omega_1 t + D_1\sin\omega_1 t) + (C_2\cos\omega_2 t + D_2\sin\omega_2 t) \\ x_B(t) &= R_1(C_1\cos\omega_1 t + D_1\sin\omega_1 t) + R_2(C_2\cos\omega_2 t + D_2\sin\omega_2 t). \end{aligned}\right\} \tag{11}$$

Now we know from Section 1 that the sinusoidal functions in the brackets can all be rewritten to give the general solution in the alternative form:

$$\left.\begin{aligned} x_A(t) &= A_1 \sin(\omega_1 t + \phi_1) + A_2 \sin(\omega_2 t + \phi_2) \\ x_B(t) &= R_1 A_1 \sin(\omega_1 t + \phi_1) + R_2 A_2 \sin(\omega_2 t + \phi_2). \end{aligned}\right\} \quad (12)$$

Both (11) and (12) are useful forms for the general solution of (10). The form given in (12) shows clearly that the general solution is a sum of two normal modes: one with angular frequency ω_1 and displacement ratio R_1; the other with angular frequency ω_2 and displacement ratio R_2. Now the sum of two sinusoids of different angular frequenciès is not a sunusoid and so by choosing different combinations of amplitudes A_1, A_2 and phases ϕ_1, ϕ_2 we can obtain an unlimited number of non-sinusoidal solutions for x_A and x_B. Thus for an arbitrary set of initial conditions we should, in general, expect a non-sinusoidal solution. The only exceptions are those particular initial conditions which satisfy the requirements of normal mode motion, and hence cut out the other normal mode angular frequency altogether. In general, however, the motion will be a non-sinusoidal combination of normal mode terms.

If we know enough about a system to be able to find the normal mode angular frequencies ω_1, ω_2 and the normal mode displacement ratios R_1, R_2 then we have enough information to write down the general solution. However, if we want to find a particular solution we still have to find four arbitrary constants (either C_1, D_1, C_2, D_2 in the case of (11) or A_1, A_2, ϕ_1, ϕ_2, in the case of (12)) and so we need four initial conditions. Here is an example.

Example 3

For the same data which we used in Exercises 3 and 6, i.e. $k_A = k_B = k$ and $m_A = m_B = m$, predict the motion of each particle when the system is released with the following initial conditions

$$x_A(0) = X,\ x_B(0) = 0 \qquad \text{and} \qquad \dot{x}_A(0) = \dot{x}_B(0) = 0$$

where X is an arbitrary number.

Solution

From Exercises 3 and 6 we know that the normal mode angular frequencies are:

$$\omega_1 = 0.62\sqrt{\frac{k}{m}} \qquad \text{and} \qquad \omega_2 = 1.62\sqrt{\frac{k}{m}},$$

and the normal mode displacement ratios are:

$$R_1 = 1.62 \qquad \text{and} \qquad R_2 = -0.62.$$

We shall use the form of the general solution given in (11):

$$\left.\begin{aligned} x_A(t) &= (C_1 \cos\omega_1 t + D_1 \sin\omega_1 t) + (C_2 \cos\omega_2 t + D_2 \sin\omega_2 t) \\ x_B(t) &= R_1(C_1 \cos\omega_1 t + D_1 \sin\omega_1 t) + R_2(C_2 \cos\omega_2 t + D_2 \sin\omega_2 t). \end{aligned}\right\} \quad (11)$$

We shall also need the velocities

$$\begin{aligned} \dot{x}_A(t) &= \omega_1(-C_1 \sin\omega_1 t + D_1 \cos\omega_1 t) \\ &\quad + \omega_2(-C_2 \sin\omega_2 t + D_2 \cos\omega_2 t) \\ \dot{x}_B(t) &= \omega_1 R_1(-C_1 \sin\omega_1 t + D_1 \cos\omega_1 t) \\ &\quad + \omega_2 R_2(-C_2 \sin\omega_2 t + D_2 \cos\omega_2 t). \end{aligned}$$

Substituting in the initial conditions we obtain

$$\begin{aligned} &C_1 + C_2 = X \\ &R_1 C_1 + R_2 C_2 = 0 \\ &\omega_1 D_1 + \omega_2 D_2 = 0 \\ &\omega_1 R_1 D_1 + \omega_2 R_2 D_2 = 0 \end{aligned}$$

which can be solved for C_1, C_2, D_1, and D_2. From the second pair of equations we obtain (since $R_1 \neq R_2$)

$$D_1 = D_2 = 0.$$

From the first pair we obtain

$$C_1 = -\frac{R_2 X}{R_1 - R_2} \simeq \frac{0.62X}{1.62 + 0.62} \simeq 0.28X$$

$$C_2 = \frac{R_1 X}{R_1 - R_2} \simeq \frac{1.62X}{1.62 + 0.62} \simeq 0.72X.$$

Substituting these values into (11), together with the values of ω_1, ω_2, R_1 and R_2, we obtain the required particular solution:

$$x_A(t) \simeq 0.28X \cos\left(0.62\sqrt{\frac{k}{m}}t\right) + 0.72X \cos\left(1.62\sqrt{\frac{k}{m}}t\right)$$

$$x_B(t) \simeq 0.45X \cos\left(0.62\sqrt{\frac{k}{m}}t\right) - 0.45X \cos\left(1.62\sqrt{\frac{k}{m}}t\right).$$

These solutions are shown plotted in Figure 6 assuming $k/m = 1$ and $X = 1$ (only part of a cycle is shown). Notice that the curves are not sinusoidal, and check that the required initial conditions are satisfied.

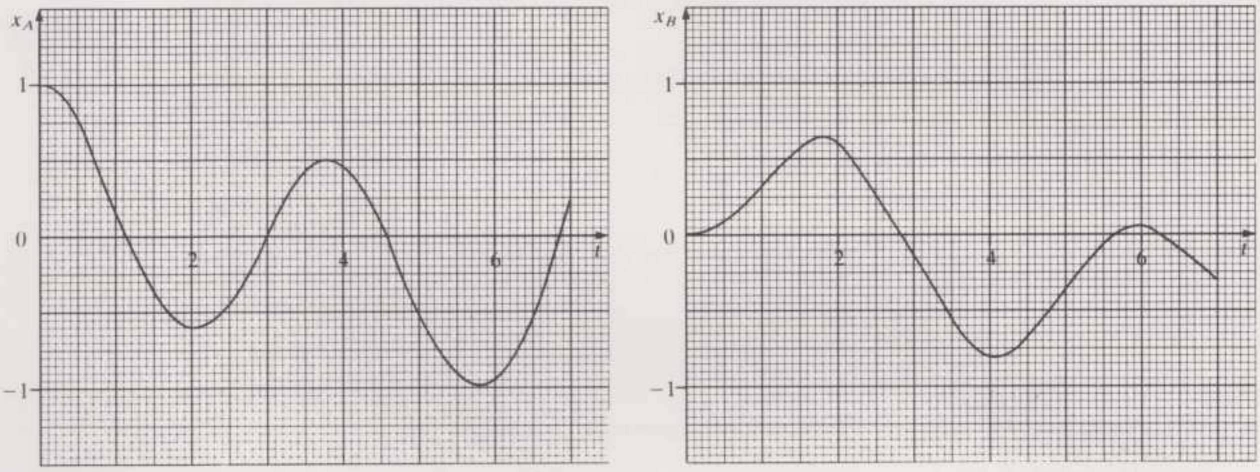


Figure 6

Exercise 7

Using the same data as in Example 3 above show that when the particles are released with the following initial conditions, both particles will perform simple harmonic motion at the same frequency:

$$x_A(0) = X,\ x_B(0) = R_1 X \quad \text{and} \quad \dot{x}_A(0) = \dot{x}_B(0) = 0.$$

Here X is an arbitrary number.

[*Solution on p. 32*]

Summary of Section 2

This section deals with undamped lumped-parameter models with two degrees of freedom. Such models have two angular frequencies at which both masses can move with simple harmonic motion. This type of motion is called a **normal mode** and the angular frequencies are called **normal mode angular frequencies**. In each normal mode the displacements of the two particles are proportional. The constant of proportionality is called the **normal mode displacement ratio**. A positive displacement ratio means that both particles move in the same direction during normal mode motion (**in-phase**); a negative displacement ratio means they move in opposite directions (**phase-opposed**).

The general free motion of an undamped lumped-parameter model with two degrees of freedom (i.e. the motion which results when the particles are released from arbitrary initial conditions) is not sinusoidal but can be shown to consist of a

superposition of appropriate proportions of two normal modes. The general expressions for the displacements x_A and x_B of the two particles take the form

$$x_A = A_1 \sin(\omega_1 t + \phi_1) + A_2 \sin(\omega_2 t + \phi_2)$$

$$x_B = R_1 A_1 \sin(\omega_1 + \phi_1) + R_2 A_2 \sin(\omega_2 t + \phi_2)$$

where ω_1 and ω_2 are the normal mode angular frequencies, R_1 and R_2 are the corresponding normal mode displacement ratios and A_1, A_2, ϕ_1 and ϕ_2 are constants which satisfy $A_1 \geqslant 0$, $A_2 \geqslant 0$, $-\pi < \phi_1 \leqslant \pi$ and $-\pi < \phi_2 \leqslant \pi$ but which are otherwise arbitrary.

3 Vibration absorption (Television Section)

3.1 Pre-broadcast notes

The programme is about undamped lumped-parameter systems with two degrees of freedom and it refers to the results of a generalized analysis as follows:

Figure 1 shows the model and the notation used in the programme. The displacements x_A and x_B are measured (as usual in this unit) from the equilibrium positions, so that the spring forces available to accelerate the particles (over and above the spring forces required to support the particles) are $k_A x_A$ and $k_B(x_B - x_A)$ respectively.

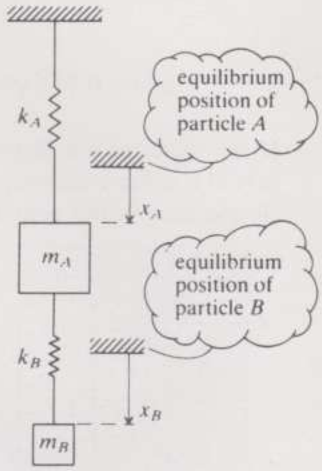


Figure 1

Taking the direction of these forces into account, it follows that the equations of motion are

$$k_B(x_B - x_A) - k_A x_A = m_A \ddot{x}_A \qquad (1)$$

$$-k_B(x_B - x_A) = m_B \ddot{x}_B. \qquad (2)$$

We can find the normal modes by substituting $\ddot{x}_A = -\omega^2 x_A$ into (1) and $\ddot{x}_B = -\omega^2 x_B$ into (2) to obtain

$$k_B(x_B - x_A) - k_A x_A = -m_A \omega^2 x_A$$

$$-k_B(x_B - x_A) = -m_B \omega^2 x_B.$$

These equations can be rewritten in the form of an eigenvector equation:

$$\begin{bmatrix} -\dfrac{(k_A + k_B)}{m_A} & \dfrac{k_B}{m_A} \\ \dfrac{k_B}{m_B} & -\dfrac{k_B}{m_B} \end{bmatrix} \begin{bmatrix} x_A \\ x_B \end{bmatrix} = -\omega^2 \begin{bmatrix} x_A \\ x_B \end{bmatrix},$$

or alternatively,

$$\begin{bmatrix} \dfrac{k_A + k_B}{m_A} - \omega^2 & -\dfrac{k_B}{m_A} \\ -\dfrac{k_B}{m_B} & \dfrac{k_B}{m_B} - \omega^2 \end{bmatrix} \begin{bmatrix} x_A \\ x_B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \qquad (3)$$

The matrix on the left hand side of this equation is quoted and discussed in the programme.

Exercise 1

Show that the characteristic equation of the model in Figure 1 is

$$\omega^4 - \left(\frac{k_B}{m_B} + \frac{k_A + k_B}{m_A}\right)\omega^2 + \frac{k_A k_B}{m_A m_B} = 0.$$

[*Solution on p. 32*]

From the characteristic equation found in Exercise 1 it is possible to find the values of ω^2, each of them corresponding to one of the two normal mode angular frequencies ω_1 and ω_2.

Now view the television programme 'Vibration absorbers'.

TV24

3.2 Vibration absorbers

The programme demonstrates free and forced vibration in undamped lumped-parameter models with two degrees of freedom, and discusses the way in which these models can predict the performance of real systems, and help with the design of vibration absorbers whose purpose is to suppress unwanted vibrations.

To illustrate free vibrations, there is a studio demonstration of the fact that, given the right initial conditions, it is possible for the free motion of a system like that in Figure 1 to take the form of simple harmonic motion of both particles, the frequency being the same for both. But it is also shown that, in general, for arbitrary initial conditions, the free motions of the particles are *not* simple harmonic.

Exercise 2

The studio demonstrations use a double spring-mass system similar to that shown in Figure 1 in which

$$m_A = 0.38(\mathrm{kg}) \qquad k_A = 15(\mathrm{N\,m^{-1}})$$

$$m_B = 0.146(\mathrm{kg}) \qquad k_B = 5.8(\mathrm{N\,m^{-1}}).$$

Estimate the normal mode angular frequencies for this system assuming it to be undamped and assuming the springs to be perfect.

[*Solution on p. 33*]

Exercise 3

For the demonstration system specified in Exercise 2, estimate a ratio for the initial displacements of the particles from their equilibrium positions so that, after release, both particles move with simple harmonic motion at the same angular frequency.

[*Solution on p. 33*]

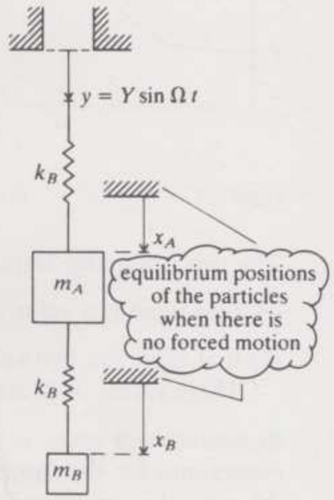


Figure 2

In the programme, forced motion is illustrated by giving the upper end of the spring in Figure 1 a vertical sinusoidal motion $y = Y\sin\Omega t$. In other words, what was the fixed end of the system is now made to move with simple harmonic motion. The studio demonstration shows that after a short time both particles settle down to a steady-state consisting of simple harmonic motion with the same angular frequency as the motion imposed on the upper end, i.e. with angular frequency Ω. The equations of motion can be derived from Figure 2 (the displacements x_A and x_B of the particles being measured from the same positions as in Figure 1):

$$-k_A(x_A - y) + k_B(x_B - x_A) = m_A\ddot{x}_A \tag{4}$$

$$-k_B(x_B - x_A) = m_B\ddot{x}_B. \tag{5}$$

Notice that, as pointed out in the programme, these two equations are very similar to Equations (1) and (2) which were obtained for the free motion of the system. The only difference is the presence of the y in Equation (4).

Remembering that $y = Y\sin\Omega t$, we can rewrite the last two equations as follows:

$$m_A\ddot{x}_A + (k_A + k_B)x_A - k_Bx_B = k_A Y\sin\Omega t \tag{6}$$

$$m_B\ddot{x}_B + k_Bx_B - k_Bx_A = 0. \tag{7}$$

We can find the steady-state solution of these simultaneous equations by following the procedure explained in Subsection 4.2 of *Unit 22*.

Exercise 4

Find the steady-state solution of Equations (6) and (7).

[*Solution on p. 33*]

From the solution to the last exercise it is clear that the amplitudes (or maximum values) of x_A and x_B are, respectively,

$$A = \frac{(k_B - m_B\Omega^2)k_A Y}{(k_A + k_B - m_A\Omega^2)(k_B - m_B\Omega^2) - k_B{}^2}$$

and

$$B = \frac{k_A k_B Y}{(k_A + k_B - m_A\Omega^2)(k_B - m_B\Omega^2) - k_B{}^2}.$$

Figure 3 shows these amplitudes plotted against Ω.

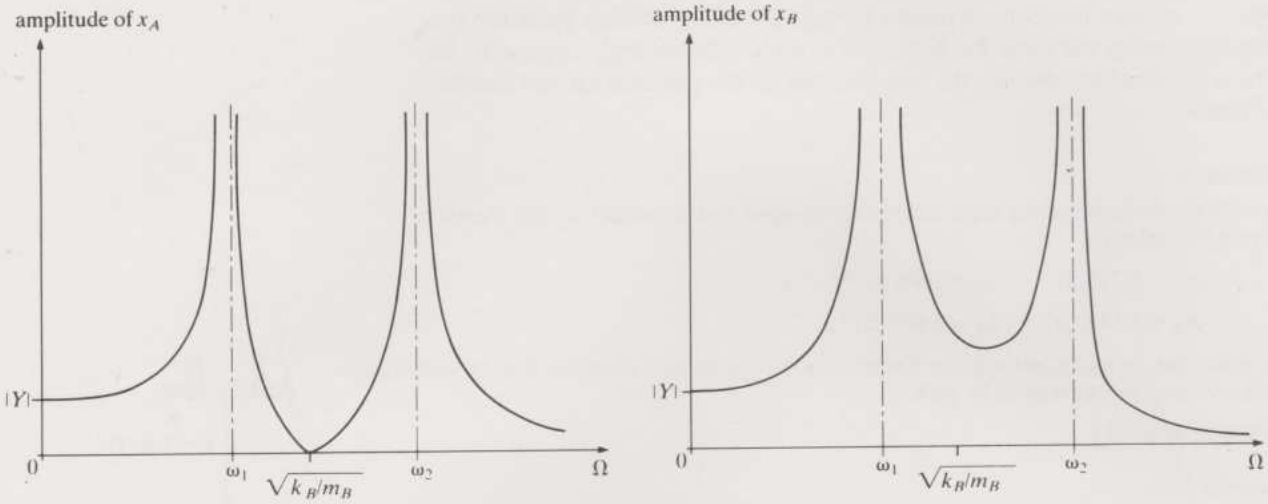


Figure 3 (a) main mass (b) absorber mass

The two noticeable features of the graphs are:

(i) that there is a value of Ω for which $x_A = 0$;

(ii) that there are two values of Ω near which both amplitudes become very large. These values of Ω are known as the **resonant angular frequencies**.

Resonance will occur at the values of Ω for which the denominator of the expressions for the amplitudes A and B is zero. (Note that both expressions have the same denominator.) The condition for resonance therefore is

$$(k_A + k_B - m_A\Omega^2)(k_B - m_B\Omega^2) - k_B{}^2 = 0,$$

i.e. $$k_A k_B - k_B m_A\Omega^2 - k_A m_B\Omega^2 - k_B m_B\Omega^2 + m_A m_B\Omega^4 = 0,$$

or $$\Omega^4 - \left(\frac{k_B}{m_B} + \frac{k_A + k_B}{m_A}\right)\Omega^2 + \frac{k_A k_B}{m_A m_B} = 0.$$

If we put Ω equal to the normal mode angular frequency ω, we get the characteristic equation of the system which you derived in Exercise 1. It follows that the normal mode angular frequencies are also the resonant angular frequencies of the system.

The value of Ω for which $x_A = 0$ can clearly be found by putting the numerator of the expression for A equal to 0. Since $k_A Y \neq 0$ (otherwise we should have no forced motion), we must have

$$k_B - m_B\Omega^2 = 0,$$

i.e. $$\Omega^2 = \frac{k_B}{m_B},$$

or

$$\boxed{\Omega = \sqrt{\frac{k_B}{m_B}}.}$$

At this angular frequency particle A is stationary even though the top end of the system continues to move according to $y = Y\sin\Omega t$, and particle B continues to vibrate, too. Under these circumstances the system is said to act as a **vibration absorber**, m_A being the **main mass** whose vibrations are absorbed (so that it is stationary) and m_B being referred to as the **absorber mass**. As you can see from the graph, fine tuning is essential for good vibration absorption; even quite a small change in the value of Ω can move the value of x_A quite a long way away from zero.

The vibration absorber designed by Southampton University to reduce the vibrations of the car deck on a Sealink ferry (now called *Earl William*, formerly *Viking II*) works on the principles I have just outlined. The vibrations are due to the action of the propellers. The mass and the stiffness of the car deck constitute the main mass m_A and the stiffness k_A. A mass of water supported on a column of air makes up the absorber mass and its spring. A schematic sketch of the arrangement is shown in Figure 4.

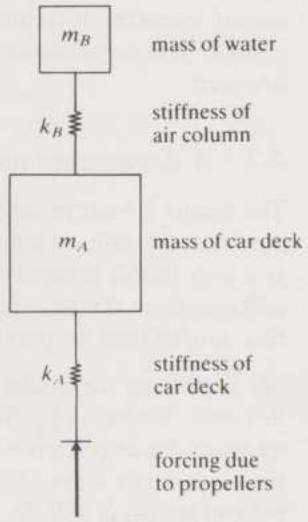


Figure 4

In the programme, Dr Hunt of Southampton University demonstrates a small version of this device and points out that it might be possible in future for the tuning to be done automatically so as to maintain a low amplitude of vibration of the car deck at all engine speeds. The present design does not have this facility but is tuned to the vibration frequency at normal speed, which is about 13 hertz (i.e. 13 cycles per second).

Summary of Section 3

The free motion of an undamped lumped-parameter model with two degrees of freedom can be expressed as a superposition of the two normal modes of vibration.

For the forced motion of the undamped lumped-parameter model with two degrees of freedom shown in Figure 2 the normal mode angular frequencies are also the **resonant angular frequencies**. The graph of steady-state amplitude against angular frequency for each particle therefore shows two resonant angular frequencies. Between these there is one angular frequency at which the amplitude of one particle (the one with mass m_A in Figure 2) is zero though the vibration of the other particle has a finite amplitude. This angular frequency is given by $\Omega = \sqrt{\dfrac{k_B}{m_B}}$ where k_B is the stiffness of the spring connecting the two particles. At this angular frequency the system is said to act as a **vibration absorber**. This principle is used to suppress unwanted vibrations, as in the case of the car deck on the ship featured in the television programme.

4 Extending the scope

4.1 About this section

In this section, I return to the discussion of normal mode motion. So far in this unit I have concentrated mainly on the simplest model which demonstrates the properties of normal modes: the undamped, two-degrees-of-freedom, lumped-parameter model shown in Figure 1 of Section 2. In the first part of this section I want to consider a physical system which is similar to this model (not least because it, too, has two degrees of freedom) but which takes a slightly different form.

The rest of this section will be devoted to coping with more than two degrees of freedom. For this, some of the mathematical methods from the unit on eigenvalues will again be very useful. I shall again stick to undamped models and I may as well say here and now that those with more than two degrees of freedom behave in much the same general way as those with just two, except that they have more normal mode angular frequencies and normal mode displacement ratios and require bigger matrices which are somewhat more laborious to handle. But that is

merely a matter of technique—the treatment of more than two degrees of freedom is really just an extension of methods we have already used: no new principles are involved.

4.2 A degenerate model

The model I want to discuss next is shown in Figure 1. You have already met this model in the unit on simple harmonic motion — the first unit on vibrations — but you may find it interesting to meet it again in a slightly different context. It is different from the two-degrees-of-freedom models that I have dealt with so far in this unit, in that no part of the model is attached to a fixed point.

Figure 1

We shall treat the model in the same way that we have treated the other models in this unit. We begin by choosing an equilibrium configuration from which to measure the displacements of the particles. Because no point of the model is fixed, there are many ways of doing this. Any configuration in which the spring has its natural length l_0 will do, for there will then be no forces acting on the particles. It does not matter which equilibrium configuration we choose so long as we pick a particular one and stick to it. Figure 2 shows a snapshot of the model at some instant after the particles have been set in motion, together with the equilibrium positions from which the displacements x_A and x_B are to be measured. The arrowheads for x_A and x_B define, as before, the positive direction for displacements, velocities, accelerations and forces. The spring force is $k(x_B - x_A)$, and so, taking the direction of forces into account, the equations of motion of the two particles are:

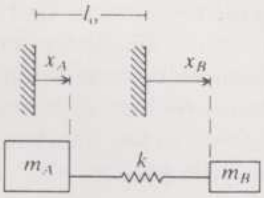


Figure 2

$$k(x_B - x_A) = m_A\ddot{x}_A$$
$$-k(x_B - x_A) = m_B\ddot{x}_B$$

which can be written in the matrix form

$$\begin{bmatrix} -\dfrac{k}{m_A} & \dfrac{k}{m_A} \\ \dfrac{k}{m_B} & -\dfrac{k}{m_B} \end{bmatrix}\begin{bmatrix} x_A \\ x_B \end{bmatrix} = \begin{bmatrix} \ddot{x}_A \\ \ddot{x}_B \end{bmatrix}.$$

For normal mode motion we put $\ddot{x}_A = -\omega^2 x_A$ and $\ddot{x}_B = -\omega^2 x_B$ to obtain the eigenvector equation

$$\begin{bmatrix} \left(-\dfrac{k}{m_A} + \omega^2\right) & \dfrac{k}{m_A} \\ \dfrac{k}{m_B} & \left(-\dfrac{k}{m_B} + \omega^2\right) \end{bmatrix}\begin{bmatrix} x_A \\ x_B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \qquad (1)$$

The characteristic equation of the model is therefore

$$\left(-\frac{k}{m_A} + \omega^2\right)\left(-\frac{k}{m_B} + \omega^2\right) - \frac{k^2}{m_A m_B} = 0,$$

or $$\omega^4 - \frac{k}{m_A}\omega^2 - \frac{k}{m_B}\omega^2 + \frac{k^2}{m_A m_B} - \frac{k^2}{m_A m_B} = 0,$$

i.e. $$\omega^4 - \left(\frac{k}{m_A} + \frac{k}{m_B}\right)\omega^2 = 0.$$

This is fairly easy to solve because there is no constant term and so we can factor out ω^2 to obtain

$$\omega^2\left(\omega^2 - \left(\frac{k}{m_A} + \frac{k}{m_B}\right)\right) = 0.$$

Hence

$$\omega_1{}^2 = 0 \qquad \text{i.e. } \omega_1 = 0$$

$$\omega_2{}^2 = \left(\frac{k}{m_A} + \frac{k}{m_B}\right) \qquad \text{i.e. } \omega_2 = \sqrt{\left(\frac{k}{m_A} + \frac{k}{m_B}\right)}.$$

A system for which one of the normal mode frequencies is zero is called a **degenerate** system. The effect that this has on the normal mode motion can easily be seen by substituting $\omega = 0$ into one of the equations in (1) above. From the top equation we get

$$-\frac{k}{m_A}x_A + \frac{k}{m_A}x_B = 0$$

so $\quad x_A = x_B.$

Thus, at any instant of the normal mode motion, both ends of the spring are displaced by the same amount and so this corresponds to a mode in which the spring is neither compressed nor extended. This means that the force on each particle is always zero and so the particles move at a constant velocity as though they were part of the same rigid body. For this reason, this normal mode is referred to as a **rigid body mode** and its existence is typical of a degenerate system. The other mode is unremarkable; you might like to work out its displacement ratio for yourself:

Exercise 1

Derive the second normal mode displacement ratio of the degenerate system in Figure 2.

[*Solution on p. 33*]

The general expression for the displacement of each particle will be similar to that which we obtained in Section 2, but with one important difference: since, as we have seen, the motion corresponding to the lower normal mode is one of constant velocity for both particles (the same velocity for both) it follows that for this normal mode

$$\dot{x}_A = \dot{x}_B = C,$$

and hence

$$x_A = x_B = Ct + D,$$

where C and D are arbitrary constants.

The motion corresponding to the second normal mode is unchanged, so that, for the general motion

$$x_A = Ct + D + A_2 \sin(\omega_2 t + \phi_2)$$
$$x_B = Ct + D + R_2 A_2 \sin(\omega_2 t + \phi_2).$$

4.3 Three degrees of freedom

As I have already said, a model with three degrees of freedom behaves in the same general way as a model with two degrees of freedom except that it has three normal mode angular frequencies and three normal mode displacement ratios. In this subsection I will consider two models with three degrees of freedom: first a degenerate model and then a non-degenerate model.

A degenerate model

This time, I will start with a numerical example. Consider the three particles and two springs shown in Figure 3.

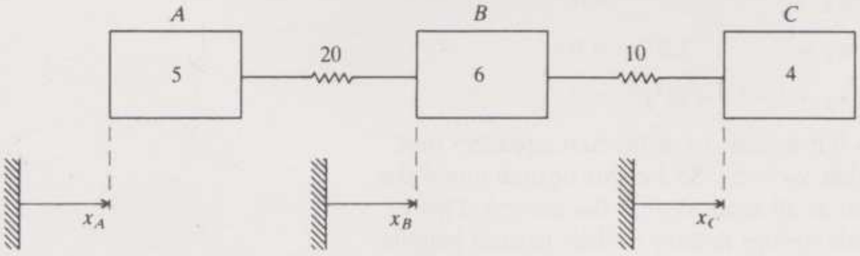


Figure 3

The masses are given in kg and the spring stiffnesses in $\mathrm{N\,m^{-1}}$, as usual. The displacements are measured from the position which each mass takes up when the whole system is in one of its equilibrium configurations (that is, a configuration in which both springs have their natural lengths). We now imagine the three particles to be set in motion, and we proceed in the usual way. First, we need the equations of motion for the three particles.

Exercise 2

Write down the equations of motion for the three particles in Figure 3.

[*Solution on p. 33*]

You can now write down the eigenvector equation for normal mode motion, by incorporating the conditions $\ddot{x}_A = -\omega^2 x_A$, $\ddot{x}_B = -\omega^2 x_B$ and $\ddot{x}_C = -\omega^2 x_C$ which specify the simple harmonic motion of the three particles.

From this eigenvector equation you can derive the characteristic equation for the model in the same way as we did for models with two degrees of freedom.

Exercise 3

Write down the eigenvector equation for normal motion and hence derive the characteristic equation of the model in Figure 3.

[*Solution on p. 33*]

So the characteristic equation of the model is

$$\omega^6 - 11.5\omega^4 + 25\omega^2 = 0$$

where ω represents the normal mode angular frequency in $\mathrm{rad\,s^{-1}}$. This is a cubic equation in ω^2, but notice again that the constant terms have cancelled, so that we can write

$$\omega^2(\omega^4 - 11.5\omega^2 + 25) = 0.$$

Hence, either $\omega^2 = 0$ or

$$(\omega^4 - 11.5\omega^2 + 25) = 0.$$

So we need only to solve a quadratic in ω^2 to find the other two normal mode angular frequencies. We have

$$\omega^2 = \frac{11.5 \pm \sqrt{132.25 - 100}}{2} = 5.75 \pm 2.84.$$

Hence

$$\begin{array}{lll} {\omega_1}^2 = 0 & \text{i.e.} & \omega_1 = 0 \\ {\omega_2}^2 = 2.91 & \text{i.e.} & \omega_2 \simeq 1.71\ (\mathrm{rad\,s^{-1}}) \\ {\omega_3}^2 = 8.59 & \text{i.e.} & \omega_3 \simeq 2.93\ (\mathrm{rad\,s^{-1}}). \end{array}$$

The lowest normal mode angular frequency is zero, and so the model is degenerate. We can quickly check that the lowest normal mode is a rigid body mode as follows.

From Exercise 3 we know that the equations for the normal mode eigenvalue problem can be written

$$\begin{aligned} (-4 + \omega^2)x_A + 4x_B \qquad\qquad &= 0 \\ 3.33x_A + (-5 + \omega^2)x_B + 1.67x_C &= 0 \\ 2.5x_B + (-2.5 + \omega^2)x_C &= 0. \end{aligned}$$

For the normal mode frequency $\omega_1 = 0$ it is clear from the first equation that $x_A = x_B$, and from the last equation that $x_B = x_C$. So for this normal mode the displacements of the particles are equal at all times during the motion. This is therefore a rigid body mode. Since both springs remain at their natural lengths, the force on each particle is zero and so all the particles move with the same constant velocity.

It is interesting to characterize this rigid body motion in terms of displacement ratios. When we were dealing with models having two particles, each normal mode had a displacement ratio x_B/x_A associated with it which related the motion of particle B to the motion of particle A. Now that we are dealing with a model having three particles, each normal mode requires an extra displacement ratio x_C/x_A to relate the motion of the third particle C to the motion of the other particles. In the case of the rigid body mode just considered $x_A = x_B = x_C$ and so both the displacement ratios x_B/x_A and x_C/x_A must be 1.

To complete the calculations, let us find the displacement ratios for the other normal modes. For the second normal mode we have $\omega^2 \simeq 2.91$ and so the equations become

$$\begin{aligned} -1.09x_A + \quad 4x_B \qquad\quad &= 0 \\ 3.33x_A - 2.09x_B + 1.67x_C &= 0 \\ 2.5x_B + 0.41x_C &= 0. \end{aligned}$$

These can be solved by Gaussian elimination as in the unit on eigenvalues; but as in our previous eigenvector calculations in this unit it is quicker to start with the first equation, which gives x_B in terms of x_A:

$$x_B = \frac{1.09}{4}x_A = 0.27x_A.$$

Then we can use the third equation to express x_C in terms of x_B and thence in terms of x_A:

$$x_C = -\frac{2.5}{0.41}x_B = -\frac{2.5}{0.41} \times 0.27x_A = -1.66x_A.$$

The middle equation can then be used as a check:

$$3.33x_A - 2.09 \times 0.27x_A + 1.67 \times (-1.66)x_A \simeq 0.$$

So, if we take $x_A = 1$, we obtain the eigenvector

$$\begin{bmatrix} 1 \\ 0.27 \\ -1.66 \end{bmatrix}.$$

Because the entry for x_A is unity, the entries 0.27 and -1.66 give the two displacement ratios x_B/x_A and x_C/x_A for the second normal mode. The change of sign shows that at any instant during the motion particles A and B are moving in the same direction (in-phase) while particle C is moving in the opposite direction (phase-opposed) to A and B.

Repeating the procedure with $\omega^2 = \omega_3^2 = 8.59$ we obtain

$$\begin{aligned} x_B &= -1.15x_A \\ x_C &= -0.41x_B = 0.47x_A. \end{aligned}$$

If we take $x_A = 1$ we obtain the eigenvector

$$\begin{bmatrix} 1 \\ -1.15 \\ 0.47 \end{bmatrix}.$$

So, in the third normal mode the displacement ratios are -1.15 and 0.47. Hence particle B is the 'odd man out', particles A and C moving in phase with each other.

A non-degenerate model

The model I shall use here is derived from the degenerate model we have just considered by adding an extra spring and using it to anchor the model to a fixed foundation, thus removing the degeneracy. The resulting model is shown in Figure 4.

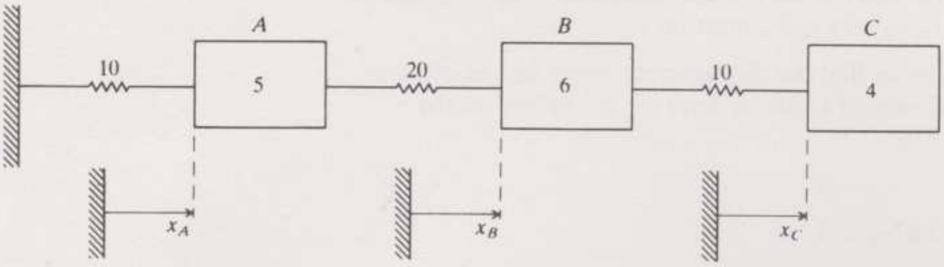


Figure 4

The addition of the third spring is reflected in a change in the equation of motion for particle A which now becomes:

$$-10x_A + 20(x_B - x_A) = 5\ddot{x}_A,$$

the other equations of motion remaining unchanged. The eigenvector equation for normal mode motion of the model in Figure 4 is therefore

$$\begin{bmatrix} -6 & 4 & 0 \\ 3.33 & -5 & 1.67 \\ 0 & 2.5 & -2.5 \end{bmatrix} \begin{bmatrix} x_A \\ x_B \\ x_C \end{bmatrix} = -\omega^2 \begin{bmatrix} x_A \\ x_B \\ x_C \end{bmatrix}, \qquad (2)$$

which is the same as the eigenvector equation obtained in the solution to Exercise 3, with the exception of the term -6 which takes the place of -4. By comparison with the solution to Exercise 3 the characteristic equation for the model is

$$(-6 + \omega^2)(8.33 - 7.5\omega^2 + \omega^4) - 13.33 \times (-2.5 + \omega^2) = 0,$$

or $\quad \omega^6 - 13.5\omega^4 + 53.33\omega^2 - 50 + 33.33 - 13.33\omega^2 = 0,$

i.e. $\quad \omega^6 - 13.5\omega^4 + 40\omega^2 - 16.67 = 0. \qquad (3)$

This equation has a constant term in it and we cannot therefore reduce it to a lower order by factoring out ω^2 as we did before. So we are faced with the problem of solving Equation (3)—a cubic in ω^2. One way to proceed is to use one of the numerical methods from the earlier units of the course. To do this it is convenient to work with the eigenvalues $\lambda = -\omega^2$ than directly with the angular frequencies ω. In terms of λ the characteristic equation (3) becomes

$$-\lambda^3 - 13.5\lambda^2 - 40\lambda - 16.67 = 0,$$

or $\quad \lambda^3 + 13.5\lambda^2 + 40\lambda + 16.67 = 0.$

This equation was solved numerically in *Unit 18* (Section 2, Exercise 1) using the Newton-Raphson method, where the roots were found to be

$$\lambda_1 \simeq -0.49707, \qquad \lambda_2 \simeq -3.5464, \qquad \lambda_3 \simeq -9.4565.$$

We could, of course, have calculated these eigenvalues using one of the numerical procedures from the unit on eigenvalues, and that would have been an equally reasonable way to proceed.

Once the eigenvalues have been found we can calculate the normal mode angular frequencies using $\lambda = -\omega^2$. We obtain (to two places of decimals)

$\omega_1{}^2 = 0.50 \quad$ i.e. $\quad \omega_1 \simeq 0.71\ (\mathrm{rad\,s^{-1}})$

$\omega_2{}^2 = 3.55 \quad$ i.e. $\quad \omega_2 \simeq 1.88\ (\mathrm{rad\,s^{-1}})$

$\omega_3{}^2 = 9.46 \quad$ i.e. $\quad \omega_3 \simeq 3.08\ (\mathrm{rad\,s^{-1}}).$

The corresponding eigenvectors (and hence the displacement ratios) can be found by substituting these values of ω in the eigenvector equation, as before.

The general case of free motion, for three degrees of freedom

Lastly, we must look at the general motion of the particles in terms of the normal modes. We first consider the non-degenerate case. Just as for the two-degrees-of-freedom model we wrote:

$$\begin{bmatrix} x_A \\ x_B \end{bmatrix} = A_1\begin{bmatrix} 1 \\ R_1 \end{bmatrix}\sin(\omega_1 t + \phi_1) + A_2\begin{bmatrix} 1 \\ R_2 \end{bmatrix}\sin(\omega_2 t + \phi_2),$$

where

ω_1 and ω_2 are the normal mode angular frequencies,

R_1 and R_2 are the normal mode displacement ratios,

A_1, A_2, ϕ_1 and ϕ_2 are constants whose values can be determined by the initial conditions,

so for a non-degenerate, three-degrees-of-freedom model we can write the general solution in the form

$$\begin{bmatrix} x_A \\ x_B \\ x_C \end{bmatrix} = A_1 \begin{bmatrix} 1 \\ R_{1B} \\ R_{1C} \end{bmatrix} \sin(\omega_1 t + \phi_1) + A_2 \begin{bmatrix} 1 \\ R_{2B} \\ R_{2C} \end{bmatrix} \sin(\omega_2 t + \phi_2)$$
$$+ A_3 \begin{bmatrix} 1 \\ R_{3B} \\ R_{3C} \end{bmatrix} \sin(\omega_3 t + \phi_3) \qquad (4)$$

As for the two-degrees-of-freedom case, this equation can be justified from the work in the unit on systems of differential equations.

where

ω_1, ω_2 and ω_3 are the normal mode angular frequencies,

R_{rB} is the displacement ratio $\dfrac{x_B}{x_A}$ corresponding to ω_r $(r = 1, 2, 3)$,

R_{rC} is the displacement ratio $\dfrac{x_C}{x_A}$ corresponding to ω_r $(r = 1, 2, 3)$.

Exercise 4

Use Equation (4) to write down an expression for the general free motion of the three particles of the model in Figure 4.

[*Solution on p. 34*]

When dealing with degenerate models the expression for the general motion given in Equation (4) must be modified. The way this is done is similar to the treatment of the two-degrees-of-freedom degenerate model in Subsection 4.2. The following example demonstrates this.

Example 1

Write down an expression for the general motion of the three particles of the degenerate model analysed at the beginning of this subsection.

Solution

For this model $\omega_1 = 0$, $\omega_2 = 1.71$ and $\omega_2 = 2.93$ and the corresponding eigenvectors are

$$\begin{bmatrix} 1 \\ R_{1B} \\ R_{1C} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ R_{2B} \\ R_{2C} \end{bmatrix} = \begin{bmatrix} 1 \\ 0.27 \\ -1.66 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ R_{3B} \\ R_{3C} \end{bmatrix} = \begin{bmatrix} 1 \\ -1.15 \\ 0.47 \end{bmatrix}$$

As in the case of the degenerate two-degrees-of-freedom model in Subsection 4.2, the motion corresponding to the lowest normal mode (for which $\omega_1 = 0$) is not sinusoidal. As I have already explained, the motion is one of constant velocity (or zero acceleration). Hence, for each of the particles,

$\ddot{x} = 0$,

i.e. $\dot{x} = C$,

so $\quad x = Ct + D,$

where C and D are constants and t stands for time. The other two normal modes are treated just as before and so the general motion is

$$\begin{bmatrix} x_A \\ x_B \\ x_C \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} (C + Dt) + \begin{bmatrix} 1 \\ 0.27 \\ -1.66 \end{bmatrix} A_2 \sin(1.71t + \phi_2)$$

$$+ \begin{bmatrix} 1 \\ -1.15 \\ 0.47 \end{bmatrix} A_3 \sin(2.93t + \phi_3).$$

4.4 Four and more degrees of freedom

By analogy with two- and three-degrees-of-freedom systems, we can see that the general motion of a four-degrees-of-freedom spring-mass system of the kind we have been considering will have the form

$$\begin{bmatrix} x_A \\ x_B \\ x_C \\ x_D \end{bmatrix} = \sum_{r=1}^{4} \begin{bmatrix} 1 \\ R_{rB} \\ R_{rC} \\ R_{rD} \end{bmatrix} A_r \sin(\omega_r t + \phi_r),$$

Degenerate models are dealt with in the usual way by replacing $A_1 \sin(\omega_1 t + \phi_1)$ by $C + Dt$.

where A_r and ϕ_r are arbitrary constants and $[1 \ R_{rB} \ R_{rC} \ R_{rD}]^T$ is an eigenvector corresponding to the eigenvalue $-\omega_r^2$. Because the first component of this eigenvector is unity the other components give the displacement ratios of the particles in the rth normal mode. The eigenvectors $[1 \ R_{rB} \ R_{rC} \ R_{rD}]^T$ and eigenvalues $\lambda = -\omega_r^2$ $(r = 1, 2, 3, 4)$ can be found by one of the numerical methods recommended in *Unit 21*. A four-degrees-of-freedom spring-mass system forms the subject of Problem 2 in Section 5.

The generalization to more than four degrees of freedom is straightforward.

Summary of Section 4

The free motion of an undamped lumped-parameter model with more than two degrees of freedom displays the same general characteristics as shown by the two-degrees-of-freedom model in Section 2 except that there are more normal modes. A three-degrees-of-freedom model has three normal modes, a four-degrees-of-freedom model has four normal modes, and so on. The general free displacements x_A, x_B and x_C of the particles of a non-degenerate three-degrees-of-freedom model for example, may be written:

$$\begin{bmatrix} x_A \\ x_B \\ x_C \end{bmatrix} = A_1 \begin{bmatrix} 1 \\ R_{1B} \\ R_{1C} \end{bmatrix} \sin(\omega_1 t + \phi_1) + A_2 \begin{bmatrix} 1 \\ R_{2B} \\ R_{2C} \end{bmatrix} \sin(\omega_2 t + \phi_2)$$

$$+ A_3 \begin{bmatrix} 1 \\ R_{3B} \\ R_{3C} \end{bmatrix} \sin(\omega_3 t + \phi_3),$$

where

$\omega_1, \omega_2, \omega_3$ are the normal mode angular frequencies,

R_{rB} is the displacement ratio $\dfrac{x_B}{x_A}$ corresponding to ω_r,

R_{rC} is the displacement ratio $\dfrac{x_C}{x_A}$ corresponding to ω_r,

$A_1, A_2, A_3, \phi_1, \phi_2, \phi_3$ are arbitrary constants.

Degenerate models (whose lowest normal mode angular frequency is zero) may have any number of degrees of freedom.

For a degenerate model with three degrees of freedom $R_{1B} = R_{1C} = 1$ and the motion corresponding to the first normal mode is one of constant velocity for all the three particles. The first term on the right-hand side of the equation for the general motion of the three particles is therefore

$$\begin{bmatrix}1\\1\\1\end{bmatrix}(C + Dt).$$

The other terms remain unchanged.

5 End of unit problems

Please refer to the study guide at the beginning of the unit.

Problem 1

Figure 1 shows two particles fixed to a length of string. The string is stretched, so that it is in tension, and, in this condition, is fixed between two rigid supports, one above the other, so that the string is vertical. Figure 1 shows the equilibrium position of the system.

By making a number of simplifying assumptions (one of which is that gravity may be neglected compared with the tension in the string), small horizontal vibrations of the two particles may be modelled as a two-degrees-of-freedom system using the following equations of motion:

$$-\frac{T}{l}x_A - \frac{T}{l}(x_A - x_B) = m_A\ddot{x}_A$$

$$\frac{T}{l}(x_A - x_B) - \frac{T}{l}x_B = m_B\ddot{x}_B$$

where T is the tension in the string, and x_A and x_B are the horizontal displacements of the particles from their equilibrium positions.

Figure 1

For the case when $m_A = m_B = 0.05\,\text{kg}$ and $l = 0.2\,\text{m}$, determine

(a) the value of T required to make the lower normal mode angular frequency equal to $300\,\text{rad}\,\text{s}^{-1}$,

(b) the corresponding value of the higher normal mode angular frequency,

(c) the corresponding normal mode displacement ratios.

For each normal mode sketch the configuration of the system when the masses are furthest from their equilibrium positions.

Predict the displacement of each mass at $t = 2$, where t is time in seconds, if the following conditions hold at $t = 0$:

$$x_A = 0, \quad x_B = 1 \quad \text{and} \quad \dot{x}_A = \dot{x}_B = 0.$$

[*Solution on p. 34*]

Problem 2

Estimate all the normal mode angular frequencies of the model shown in Figure 2, in terms of the ratio $\frac{k}{m}$, given that:

(a) the characteristic equation is

$$\lambda(\lambda^3 + 6\lambda^2 + 10\lambda + 4) = 0$$

where $\lambda = -\omega^2 m/k$;

(b) one of the roots of the characteristic equation is -2.

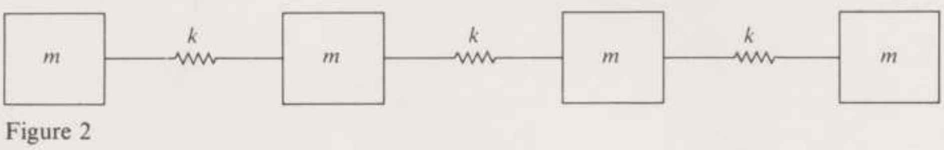


Figure 2

Derive the displacement ratios corresponding to the highest normal mode angular frequency.

[*Solution on p. 35*]

Problem 3

An example of the use of an undamped absorber is furnished by the motor/generator set shown in Figure 3. It is used to compensate for any slight unbalance of the shaft which would set up a periodic force and cause vibràtion. Notice that the total absorber mass is made up of two equal parts, each attached by a leaf spring at opposite ends of the machine. The two leaf springs have equal stiffnesses.

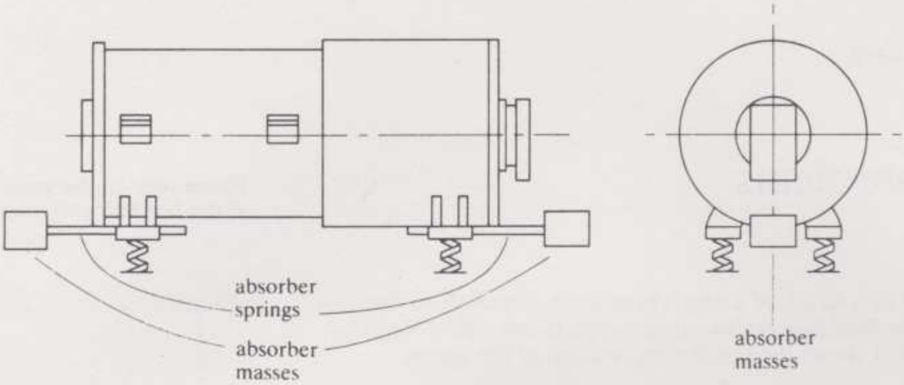


Figure 3

This motor/generator set with its absorber was the source of the data on which Exercise 4 in Section 2 was based. That exercise was concerned only with the vertical vibrations of the machine and so is this problem.

In that exercise the main mass was 1.67×10^3 kg and it was attached to the foundation by springs of total stiffness $12 \times 10^6\,\mathrm{N\,m^{-1}}$. Each absorber mass was 83.5 kg and the stiffness of the spring by which it was attached to the main mass was $2.95 \times 10^6\,\mathrm{N\,m^{-1}}$. The main mass is subject to a sinusoidally varying force which is due to the unbalanced shaft.

(i) At what angular frequency will the main mass remain at rest?

(ii) During steady running, with the absorber keeping the main mass at rest, the total up and down movement of each absorber mass is observed to be 0.002 m. Calculate the maximum vertical force on the main mass due to the unbalance of the shaft.

[*Solution on p. 36*]

Appendix 1: Solutions to the exercises

Solutions to the exercises in Section 1

1. Since $r = 0$, we can apply Newton's second law to Figure 1 as if the dashpot were not present. If $x = x_0$ when the spring is unstretched, then the equation of motion is

$$m\ddot{x} = -k(x - x_0)$$

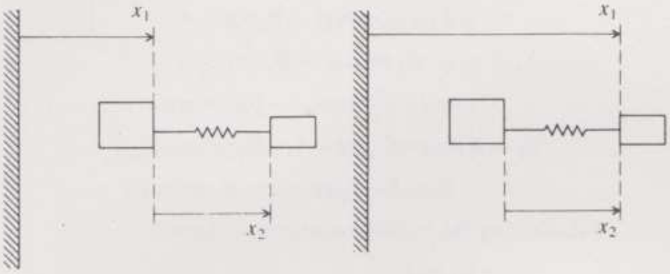
or $\quad m\ddot{x} + kx = kx_0.$

From *Unit* 7 we can recognize this as the equation for simple harmonic motion, the angular frequency being given by

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{2 \times 10^6}{0.1}} \simeq 4.47 \times 10^3.$$

The angular frequency (often called the 'natural' angular frequency) is therefore 4.47×10^3 radians per second.

2. The figures below show two more ways of specifying the configuration of the system.



Since all these representations use two coordinates the system clearly has two degrees of freedom.

3. The model in Figure 3 has seven degrees of freedom.

4. **(a)**, **(c)** and **(d)** all have two degrees of freedom. **(b)** has one degree of freedom. **(e)** and **(f)** both have three degrees of freedom. **(g)** has four degrees of freedom.

Notice that the number of degrees of freedom does not depend on the number of forces acting on each particle—it is a purely geometrical notion: the smallest number of displacements from any fixed datum (arbitrarily specified) that is required to describe the configuration of the assembly at any moment during the motion (irrespective of what forces may be acting). For example, the two rigidly connected particles in (b) need only one coordinate to specify their joint position (in fact, they are equivalent to just one particle); and so this model has only one degree of freedom.

5. The particle has 3 degrees of freedom. If you look back to what I said about Figure 4 in Subsection 1.3 you will realize that we now need an extra coordinate to allow for motion into and out of the paper.

Solutions to the exercises in Section 2

1. The equilibrium position corresponds to zero spring force, so that both springs have their natural lengths. It follows that the 0.1 kg mass is 0.6 m from the datum line and the 0.2 kg mass is 1.2 m from the datum line.

2. Here equilibrium must cater for the support of both particles so that the force in the lower spring (attached to the 0.2 kg mass) must equal $0.2 \times 9.81 \simeq 1.96$ N and the force in the upper spring must equal $(0.2 + 0.1) \times 9.81 \simeq 2.94$ N.

The extension (beyond the natural length) of the upper spring is, therefore, $\dfrac{2.94}{75} = 0.039$ m so that the upper particle is $0.039 + 0.6 = 0.639$ m below the support. Similarly the extension of the lower spring is $\dfrac{1.96}{75} \simeq 0.026$ m so that the lower particle is $0.639 + 0.6 + 0.026 = 1.265$ m below the support.

3. The equations of motion are:

$$k(x_B - x_A) - kx_A = m\ddot{x}_A$$

$$-k(x_B - x_A) = m\ddot{x}_B$$

and so the eigenvector equation for normal mode motion is

$$\begin{bmatrix} -\dfrac{2k}{m} & \dfrac{k}{m} \\ \dfrac{k}{m} & -\dfrac{k}{m} \end{bmatrix} \begin{bmatrix} x_A \\ x_B \end{bmatrix} = -\omega^2 \begin{bmatrix} x_A \\ x_B \end{bmatrix}.$$

The characteristic equation for the model is therefore

$$\begin{vmatrix} \left(\omega^2 - \dfrac{2k}{m}\right) & \dfrac{k}{m} \\ \dfrac{k}{m} & \left(\omega^2 - \dfrac{k}{m}\right) \end{vmatrix} = 0,$$

i.e. $\quad \left(\omega^2 - \dfrac{2k}{m}\right)\left(\omega^2 - \dfrac{k}{m}\right) - \dfrac{k^2}{m^2} = 0$

or $\quad \omega^4 - \dfrac{3k}{m}\omega^2 + \dfrac{k^2}{m^2} = 0.$

Solving this equation for ω^2 we obtain

$$\omega^2 = \frac{\dfrac{3k}{m} \pm \sqrt{\dfrac{9k^2}{m^2} - \dfrac{4k^2}{m^2}}}{2} = \left(\frac{3}{2} \pm \frac{\sqrt{5}}{2}\right)\frac{k}{m}.$$

Denoting the lower of the two values of ω by ω_1 and the higher by ω_2, we have

$$\omega_1{}^2 = \left(\frac{3}{2} - \frac{\sqrt{5}}{2}\right)\frac{k}{m} = 0.38\frac{k}{m}$$

i.e. $\quad \omega_1 \simeq 0.62\sqrt{\dfrac{k}{m}}\,(\text{rad s}^{-1})$

$$\omega_2{}^2 = \left(\frac{3}{2} + \frac{\sqrt{5}}{2}\right)\frac{k}{m} = 2.62\frac{k}{m}$$

i.e. $\quad \omega_2 \simeq 1.62\sqrt{\dfrac{k}{m}}\,(\text{rad s}^{-1}).$

4. The equations of motion are:

$$5.9 \times 10^6(x_B - x_A) - 12 \times 10^6 x_A = 1.67 \times 10^3 \ddot{x}_A,$$

$$-5.9 \times 10^6(x_B - x_A) = 167\ddot{x}_B,$$

and so the eigenvector equation for normal mode motion is

$$\begin{bmatrix} -\dfrac{17.9 \times 10^6}{1.67 \times 10^3} & \dfrac{5.9 \times 10^6}{1.67 \times 10^3} \\ \dfrac{5.9 \times 10^6}{167} & -\dfrac{5.9 \times 10^6}{167} \end{bmatrix} \begin{bmatrix} x_A \\ x_B \end{bmatrix} \doteq -\omega^2 \begin{bmatrix} x_A \\ x_B \end{bmatrix}.$$

The characteristic equation for the model is therefore

$$\begin{vmatrix} (\omega^2 - 10.72 \times 10^3) & 3.53 \times 10^3 \\ 35.33 \times 10^3 & (\omega^2 - 35.33 \times 10^3) \end{vmatrix} = 0,$$

i.e. $(\omega^2 - 10.72 \times 10^3)(\omega^2 - 35.33 \times 10^3) - 124.82 \times 10^6 = 0,$

or $\omega^4 - 46.05\omega^2 \times 10^3 + 253.86 \times 10^6 = 0.$

Solving this equation for ω^2 we obtain

$$\omega^2 = \frac{46.05 \times 10^3 \pm \sqrt{2120.41 \times 10^6 - 1015.45 \times 10^6}}{2}$$

$$= 23.02 \times 10^3 \pm 16.62 \times 10^3.$$

So we have

$$\omega_1{}^2 = 6.40 \times 10^3 \quad \text{i.e.} \quad \omega_1 \simeq 80 \text{ (rad s}^{-1})$$

$$\omega_2{}^2 = 39.64 \times 10^3 \quad \text{i.e.} \quad \omega_2 \simeq 199 \text{ (rad s}^{-1}).$$

5. In Exercise 4 we considered the eigenvector equation

$$\begin{bmatrix} -10.72 \times 10^3 & 3.53 \times 10^3 \\ 35.33 \times 10^3 & -35.33 \times 10^3 \end{bmatrix} \begin{bmatrix} x_A \\ x_B \end{bmatrix} = -\omega^2 \begin{bmatrix} x_A \\ x_B \end{bmatrix}$$

and found two normal mode angular frequencies ω_1 and ω_2, where $\omega_1{}^2 = 6.40 \times 10^3$ and $\omega_2{}^2 = 39.64 \times 10^3$. The corresponding eigenvectors can be found by substituting these values of ω^2 into

$$\begin{bmatrix} -10.72 \times 10^3 + \omega^2 & 3.53 \times 10^3 \\ 35.33 \times 10^3 & -35.33 \times 10^3 + \omega^2 \end{bmatrix} \begin{bmatrix} x_A \\ x_B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and solving for x_A and x_B.

In the case $\omega^2 = 6.40 \times 10^3$ we obtain the equations

$$-4.32 \times 10^3 x_A + 3.53 \times 10^3 x_B = 0$$

$$35.33 \times 10^3 x_A - 28.93 \times 10^3 x_B = 0$$

both of which give $x_B \simeq 1.22 x_A$. Putting $x_A = 1$ gives the eigenvector $[1 \;\; 1.22]^T$, so the first normal mode displacement ratio is 1.22.

In the case $\omega^2 = 39.64 \times 10^3$ we obtain the equations

$$28.92 \times 10^3 x_A + 3.53 \times 10^3 x_B = 0$$

$$35.33 \times 10^3 x_A + 4.31 \times 10^3 x_B = 0$$

both of which give $x_B \simeq -8.2 x_A$. Putting $x_A = 1$ gives the eigenvector $[1 \;\; -8.2]^T$, so the second normal mode displacement ratio is -8.2.

6. In Exercise 3 we considered the eigenvector equation

$$\begin{bmatrix} -\dfrac{2k}{m} & \dfrac{k}{m} \\ \dfrac{k}{m} & -\dfrac{k}{m} \end{bmatrix} \begin{bmatrix} x_A \\ x_B \end{bmatrix} = -\omega^2 \begin{bmatrix} x_A \\ x_B \end{bmatrix}$$

and found two normal mode frequencies ω_1 and ω_2, where $\omega_1{}^2 = 0.38\dfrac{k}{m}$ and $\omega_2{}^2 = 2.62\dfrac{k}{m}$. The corresponding eigenvectors can be found by substituting these values of ω^2 into

$$\begin{bmatrix} -\dfrac{2k}{m} + \omega^2 & \dfrac{k}{m} \\ \dfrac{k}{m} & -\dfrac{k}{m} + \omega^2 \end{bmatrix} \begin{bmatrix} x_A \\ x_B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and solving for x_A and x_B.

In the case $\omega^2 = 0.38\dfrac{k}{m}$ we obtain the equations

$$-1.62\frac{k}{m}x_A + \frac{k}{m}x_B = 0$$

$$\frac{k}{m}x_A - 0.62\frac{k}{m}x_B = 0$$

both of which give $x_B = 1.62 x_A$. Putting $x_A = 1$ gives the eigenvector $[1 \;\; 1.62]^T$, so the first normal mode displacement ratio is 1.62.

In the case $\omega^2 = 2.62\dfrac{k}{m}$ we obtain the equations

$$0.62\frac{k}{m}x_A + \frac{k}{m}x_B = 0$$

$$\frac{k}{m}x_A + 1.62\frac{k}{m}x_B = 0$$

both of which give $x_B = -0.62 x_A$. Putting $x_A = 1$ gives the eigenvector $[1 \;\; -0.62]^T$ so the second normal mode displacement ratio is -0.62.

7. We can use the equations from Example 3:

$$\left.\begin{aligned} x_A &= C_1 \cos \omega_1 t + D_1 \sin \omega_1 t \\ &\quad + C_2 \cos \omega_2 t + D_2 \sin \omega_2 t \\ x_B &= R_1(C_1 \cos \omega_1 t + D_1 \sin \omega_1 t) \\ &\quad + R_2(C_2 \cos \omega_2 t + D_2 \sin \omega_2 t) \end{aligned}\right\} \qquad \text{(S1)}$$

$$\dot{x}_A = \omega_1(-C_1 \sin \omega_1 t + D_1 \cos \omega_1 t) + \omega_2(-C_2 \sin \omega_2 t + D_2 \cos \omega_2 t)$$

$$\dot{x}_B = R_1\omega_1(-C_1 \sin \omega_1 t + D_1 \cos \omega_1 t) + R_2\omega_2(-C_2 \sin \omega_2 t + D_2 \cos \omega_2 t).$$

Substituting the initial conditions we have:

$$C_1 + C_2 = X$$

$$R_1 C_1 + R_2 C_2 = R_1 X$$

$$D_1 \omega_1 + D_2 \omega_2 = 0$$

$$R_1 D_1 \omega_1 + R_2 D_2 \omega_2 = 0.$$

From the second pair of equations we obtain, as in Example 3,

$$D_1 = D_2 = 0.$$

From the first pair we get

$$C_1 = X, \qquad C_2 = 0.$$

Hence, substituting in (S1) we find:

$$x_A = C_1 \cos \omega_1 t = X \cos\left(0.62 \sqrt{\frac{k}{m}}\, t\right)$$

$$x_B = R_1 C_1 \cos \omega_1 t = R_1 X \cos\left(0.62 \sqrt{\frac{k}{m}}\, t\right).$$

This is the first normal mode motion for the system and both particles perform simple harmonic motion at the same frequency.

Solutions to the exercises in Section 3

1. The characteristic equation is obtained by equating the determinant of the matrix on the left hand side of Equation (3) to zero, as follows:

$$\begin{vmatrix} \dfrac{k_A + k_B}{m_A} - \omega^2 & -\dfrac{k_B}{m_A} \\ -\dfrac{k_B}{m_B} & \dfrac{k_B}{m_B} - \omega^2 \end{vmatrix} = 0$$

That is,

$$\left(\frac{k_A + k_B}{m_A} - \omega^2\right)\left(\frac{k_B}{m_B} - \omega^2\right) - \frac{k_B{}^2}{m_A m_B} = 0,$$

or $\quad \omega^4 - \left(\frac{k_B}{m_B} + \frac{k_A + k_B}{m_A}\right)\omega^2 + \frac{k_A k_B}{m_A m_B} = 0$

as required.

2. Using the characteristic equation found in Exercise 1 and substituting the given values for masses and spring stiffnesses, we get

$$\omega^4 - \left(\frac{5.8}{0.146} + \frac{20.8}{0.38}\right)\omega^2 + \frac{15 \times 5.8}{0.38 \times 0.146} = 0$$

i.e. $\quad \omega^4 - 94.46\omega^2 + 1568.13 = 0.$

Solving this for ω^2 gives

$$\omega^2 = \frac{94.46 \pm \sqrt{8923.23 - 6272.53}}{2} = 47.23 \pm 25.74.$$

Hence

$$\omega_1{}^2 = 21.49 \quad \text{i.e.} \quad \omega_1 \simeq 4.64\,(\text{rad s}^{-1})$$
$$\omega_2{}^2 = 72.97 \quad \text{i.e.} \quad \omega_2 \simeq 8.54\,(\text{rad s}^{-1}).$$

3. If the initial displacements from the equilibrium configuration are in the same ratio as one of the normal mode displacement ratios the system will have normal mode motion, as required. So there are two possible answers — one for each normal mode.

The displacement ratios can be calculated from either of the equations which make up the matrix equation (3). The bottom equation is

$$-\frac{k_B}{m_B}x_A + \left(\frac{k_B}{m_B} - \omega^2\right)x_B = 0,$$

which on substituting the values of k_A, k_B, m_A and m_B from Exercise 2 becomes

$$-\frac{5.8}{0.146}x_A + \left(\frac{5.8}{0.146} - \omega^2\right)x_B = 0,$$

i.e. $\quad -39.73x_A + (39.73 - \omega^2)x_B = 0.$

Hence

$$\frac{x_B}{x_A} = \frac{39.73}{39.73 - \omega^2}.$$

This can take two possible values, one corresponding to each value of ω^2. From Exercise 2 we have $\omega_1{}^2 = 21.49$ and $\omega_2{}^2 = 72.97$ so for the lower normal mode angular frequency the required ratio for the displacements is

$$\frac{x_B}{x_A} = \frac{39.73}{39.73 - 21.49} = \frac{39.73}{18.24} \simeq 2.18;$$

and for the higher normal mode frequency it is

$$\frac{x_B}{x_A} = \frac{39.73}{39.73 - 72.97} = \frac{39.73}{-33.25} \simeq -1.19.$$

A positive value indicates in-phase free motion of the particles, a negative value phase-opposed free motion. This tallies with the observation of the behaviour of the studio demonstration system.

4. Following the procedure described in Subsection 4.2 of *Unit 22* we look for a solution of the form

$$\begin{bmatrix} x_A \\ x_B \end{bmatrix} = \text{Re}\left(\begin{bmatrix} a_A \\ a_B \end{bmatrix} e^{i\Omega t}\right),$$

which leads to the simultaneous linear equations

$$-m_A\Omega^2 a_A + (k_A + k_B)a_A - k_B a_B = -k_A Yi \qquad \text{(S2)}$$
$$-m_B\Omega^2 a_B + k_B a_B - k_B a_A = 0. \qquad \text{(S3)}$$

From (S3) we have

$$a_B = \left(\frac{k_B}{k_B - m_B\Omega^2}\right)a_A. \qquad \text{(S4)}$$

Substituting in (S2) we have

$$\left(-m_A\Omega^2 + (k_A + k_B) - \frac{k_B{}^2}{k_B - m_B\Omega^2}\right)a_A = -k_A Yi,$$

or $\quad \left(\frac{(k_A + k_B - m_A\Omega^2)(k_B - m_B\Omega^2) - k_B{}^2}{k_B - m_B\Omega^2}\right)a_A = -k_A Yi,$

i.e. $\quad a_A = \frac{-k_A Yi(k_B - m_B\Omega^2)}{(k_A + k_B - m_A\Omega^2)(k_B - m_B\Omega^2) - k_B{}^2}.$

Hence

$$x_A = \frac{(k_B - m_B\Omega^2)k_A Y \sin\Omega t}{(k_A + k_B - m_A\Omega^2)(k_B - m_B\Omega^2) - k_B{}^2}$$

and so, from Equation (S4) it follows that

$$x_B = \frac{k_B k_A Y \sin\Omega t}{(k_A + k_B - m_A\Omega^2)(k_B - m_B\Omega^2) - k_B{}^2}.$$

These results tally with the observation that both particles move with simple harmonic motion of angular frequency Ω.

Solutions to the exercises in Section 4

1. Putting $\omega^2 = \omega_2{}^2 = \left(\frac{k}{m_A} + \frac{k}{m_B}\right)$ into the top line of the matrix equation (1), we have

$$\left(-\frac{k}{m_A} + \left(\frac{k}{m_A} + \frac{k}{m_B}\right)\right)x_A + \frac{k}{m_A}x_B = 0,$$

so $\quad \frac{x_A}{m_B} + \frac{x_B}{m_A} = 0,$

i.e. $\quad x_A = -\frac{m_B}{m_A}x_B.$

Thus, putting $x_A = 1$ we obtain the eigenvector

$$\begin{bmatrix} 1 \\ -\frac{m_B}{m_A} \end{bmatrix},$$

and so the required displacement ratio is $-\frac{m_B}{m_A}$.

2. The equations of motion are as follows.

$$20(x_B - x_A) = 5\ddot{x}_A \quad \text{(particle } A)$$
$$-20(x_B - x_A) + 10(x_C - x_B) = 6\ddot{x}_B \quad \text{(particle } B)$$
$$-10(x_C - x_B) = 4\ddot{x}_C \quad \text{(particle } C)$$

3. The eigenvector equation is

$$\begin{bmatrix} -4 & 4 & 0 \\ \frac{20}{6} & -5 & \frac{10}{6} \\ 0 & \frac{10}{4} & -\frac{10}{4} \end{bmatrix} \begin{bmatrix} x_A \\ x_B \\ x_C \end{bmatrix} = -\omega^2 \begin{bmatrix} x_A \\ x_B \\ x_C \end{bmatrix}$$

or alternatively,

$$\begin{bmatrix} (-4 + \omega^2) & 4 & 0 \\ 3.33 & (-5 + \omega^2) & 1.67 \\ 0 & 2.5 & (-2.5 + \omega^2) \end{bmatrix} \begin{bmatrix} x_A \\ x_B \\ x_C \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The characteristic equation of the model is obtained, as usual, by evaluating the determinant of the matrix on the left hand side and equating the result to zero. That is

$$(-4+\omega^2)((-5+\omega^2)(-2.5+\omega^2)-(1.67\times 2.5))$$
$$-4\times 3.33\times(-2.5+\omega^2)=0$$

so $\quad (-4+\omega^2)(12.5-7.5\omega^2+\omega^4-4.17)$
$$-13.33\times(-2.5+\omega^2)=0$$

or $\quad (\omega^6-11.5\omega^4+38.33\omega^2-33.33)$
$$-(-33.33+13.33\omega^2)=0$$

i.e. $\quad \omega^6-11.5\omega^4+25\omega^2=0.$

This is the characteristic equation of the model.

4. First, we need to find the eigenveetors corresponding to each of the normal mode angular frequencies. We can do this by substituting each of the normal mode angular frequencies into the first and last of the three equations which constitute the matrix equation (2).

First normal mode: $\omega_1{}^2=0.50$.

From the first equation

$$-6x_A+4x_B=-0.50x_A$$

i.e. $\quad 4x_B=5.50x_A$

so $\quad \dfrac{x_B}{x_A}=\dfrac{5.50}{4}=1.38.$

From the third equation

$$2.5x_B-2.5x_C=-0.50x_C$$

i.e. $\quad 2.5x_B=2.00x_C$

so $\quad \dfrac{x_C}{x_B}=\dfrac{2.5}{2.00}=1.25.$

Thus, putting $x_A=1$ we have $x_B=1.38$ and $x_C=1.25\times 1.38=1.72$, and so, the lowest normal mode eigenvector is $[1\ \ 1.38\ \ 1.72]^T$.

Second normal mode: $\omega_2{}^2=3.55$.

From the first equation

$$-6x_A+4x_B=-3.55x_A,$$

i.e. $\quad 4x_B=2.45x_A,$

so $\quad \dfrac{x_B}{x_A}=\dfrac{2.45}{4}=0.61.$

From the third equation

$$2.5x_B-2.5x_C=-3.55x_C$$

i.e. $\quad 2.5x_B=-1.05x_C$

so $\quad \dfrac{x_C}{x_B}=-\dfrac{2.5}{1.05}=-2.39.$

Thus, putting $x_A=1$, we have $x_B=0.61$ and $x_C=-2.39\times 0.61=-1.47$, and so the second normal mode eigenvector is $[1\ \ 0.61\ \ -1.47]^T$.

Third normal mode: $\omega_3{}^2=9.46$.

From the first equation

$$-6x_A+4x_B=-9.45x_A,$$

i.e. $\quad 4x_B=-3.46x_A,$

so $\quad \dfrac{x_B}{x_A}=-\dfrac{3.46}{4}=-0.86.$

From the third equation

$$2.5x_B-2.5x_C=-9.45x_C,$$

i.e. $\quad 2.5x_B=-6.96x_C,$

so $\quad \dfrac{x_C}{x_B}=-\dfrac{2.5}{6.95}=-0.36.$

Thus, putting $x_A=1$, we have $x_B=-0.86$ and $x_C=(-0.36)\times(-0.86)=0.31$, and so the highest normal mode eigenvector is $[1\ \ -0.86\ \ 0.31]^T$.

We can now write the general expression for the free motion of the three particles in terms of the eigenvectors and the normal mode angular frequencies by substituting into Equation (4):

$$\begin{bmatrix}x_A\\x_B\\x_C\end{bmatrix}=A_1\begin{bmatrix}1\\1.38\\1.72\end{bmatrix}\sin(0.71t+\phi_1)$$
$$+A_2\begin{bmatrix}1\\0.61\\-1.47\end{bmatrix}\sin(1.88t+\phi_2)$$
$$+A_3\begin{bmatrix}1\\-0.86\\0.31\end{bmatrix}\sin(3.08t+\phi_3).$$

Appendix 2: Solutions to the problems

Solutions to the end of unit problems

1. The given equations of motion are

$$-\frac{T}{l}x_A-\frac{T}{l}(x_A-x_B)=m\ddot{x}_A$$

$$\frac{T}{l}(x_A-x_B)-\frac{T}{l}x_B=m\ddot{x}_B$$

where $m=m_A=m_B=0.05$ and $l=0.2$.

For normal mode motion $\ddot{x}_A=-\omega^2x_A$ and $\ddot{x}_B=-\omega^2x_B$ so that

$$\begin{bmatrix}2\dfrac{T}{ml}-\omega^2 & -\dfrac{T}{ml}\\ -\dfrac{T}{ml} & 2\dfrac{T}{ml}-\omega^2\end{bmatrix}\begin{bmatrix}x_A\\x_B\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}. \qquad \text{(S1)}$$

The characteristic equation for the model is therefore

$$\begin{vmatrix}2\dfrac{T}{ml}-\omega^2 & -\dfrac{T}{ml}\\ -\dfrac{T}{ml} & 2\dfrac{T}{ml}-\omega^2\end{vmatrix}=0.$$

We can simplify this equation by putting $\mu=\dfrac{T}{ml}$ to obtain

$$(2\mu-\omega^2)(2\mu-\omega^2)-\mu^2=0,$$

i.e. $\quad 4\mu^2-4\mu\omega^2+\omega^4-\mu^2=0,$

or $\quad \omega^4-4\mu\omega^2+3\mu^2=0,$

so $\quad (\omega^2-3\mu)(\omega^2-\mu)=0.$

Hence, for the normal mode angular frequencies we have

$$\omega_1{}^2=\mu=\frac{T}{ml}\qquad \text{i.e.}\qquad \omega_1=\sqrt{\frac{T}{ml}}$$

$$\omega_2{}^2 = 3\mu = 3\frac{T}{ml} \quad \text{i.e.} \quad \omega_2 = \sqrt{\frac{3T}{ml}}.$$

(a) Substituting $m = 0.05$ and $l = 0.2$ and putting $\omega_1 = 300$ we obtain

$$\frac{T}{0.05 \times 0.2} = (300)^2,$$

$$\text{i.e.} \quad T = (300)^2 \times 0.05 \times 0.2 = 900,$$

so the required tension is 900 newtons.

(b) We know that $\omega_2 = \sqrt{3}\omega_1$ so in this case

$$\omega_2 = \sqrt{3} \times 300 \simeq 519.62,$$

so the larger normal mode angular frequency is about 519.62 rad s^{-1}.

(c) To find the two normal mode displacement ratios we substitute each of the two values of normal mode angular frequency in one of the two linear equations which make up the matrix equation (S1). Using the top equation and starting with $\omega_1{}^2 = \dfrac{T}{ml}$, we have

$$\left(2\frac{T}{ml} - \frac{T}{ml}\right)x_A - \frac{T}{ml}x_B = 0,$$

$$\text{i.e.} \quad \frac{T}{ml}(x_A - x_B) = 0.$$

Hence $x_B = x_A$, so the first normal mode displacement ratio $R_1 = 1$, or in eigenvector form $[1 \;\; R_1]^T = [1 \;\; 1]^T$.

This means that the particles will move in phase with equal amplitudes.

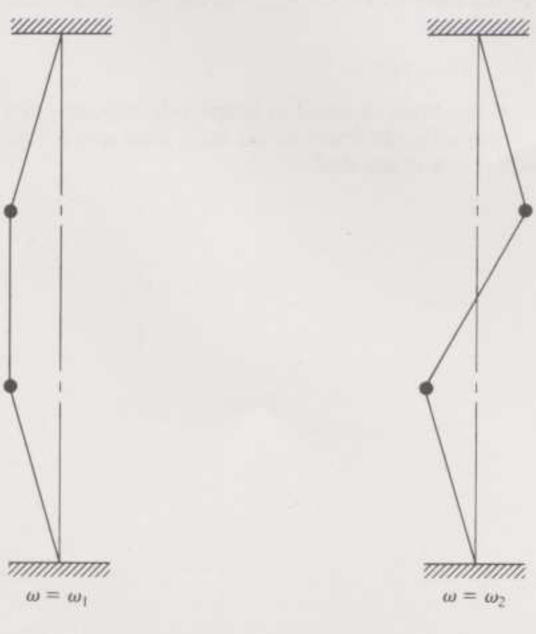
Next, putting $\omega_2{}^2 = \dfrac{3T}{ml}$ into the same equation we obtain

$$\left(2\frac{T}{ml} - \frac{3T}{ml}\right)x_A - \frac{T}{ml}x_B = 0$$

$$\text{i.e.} \quad -\frac{T}{ml}(x_A + x_B) = 0.$$

Hence $x_B = -x_A$, so the second normal mode displacement ratio $R_2 = -1$, or in eigenvector form $[1 \;\; R_2]^T = [1 \;\; -1]^T$. So the particles again move with equal amplitudes but they will be phase-opposed.

When the particles are furthest from their equilibrium positions their positions are as sketched below.



We have just seen that

$$\omega_1 = 300 \quad \text{and} \quad \omega_2 \simeq 519.62$$

with corresponding eigenvectors

$$\begin{bmatrix} 1 \\ R_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ R_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

so from Equation (11) of Section 2 the general solution is

$$\left.\begin{aligned} x_A &= C_1 \cos 300t + D_1 \sin 300t \\ &\quad + C_2 \cos 519.62t + D_2 \sin 519.62t \\ x_B &= C_1 \cos 300t + D_1 \sin 300t \\ &\quad - C_2 \cos 519.62t - D_2 \sin 519.62t. \end{aligned}\right\} \quad \text{(S2)}$$

We shall also need the velocities:

$$\begin{aligned} \dot{x}_A &= 300(-C_1 \sin 300t + D_1 \cos 300t) \\ &\quad + 519.62(-C_2 \sin 519.62t + D_2 \cos 519.62t) \\ \dot{x}_B &= 300(-C_1 \sin 300t + D_1 \cos 300t) \\ &\quad + 519.62(C_2 \sin 519.62t - D_2 \cos 519.62t). \end{aligned}$$

Substituting in the initial conditions $x_A(0) = 0$, $x_B(0) = 1$ and $\dot{x}_A(0) = \dot{x}_B(0) = 0$ gives

$$C_1 + C_2 = 0$$

$$C_1 - C_2 = 1$$

$$300D_1 + 519.62D_2 = 0$$

$$300D_1 - 519.62D_2 = 0$$

which can be solved for C_1, C_2, D_1 and D_2. From the second pair of equations we obtain

$$D_1 = D_2 = 0.$$

From the first pair we obtain

$$C_1 = \frac{1}{2}, \quad C_2 = -\frac{1}{2}.$$

Substituting these values into (S2) gives the particular solution:

$$x_A = \tfrac{1}{2}\cos 300t - \tfrac{1}{2}\cos 519.62t$$

$$x_B = \tfrac{1}{2}\cos 300t + \tfrac{1}{2}\cos 519.62t.$$

Thus when $t = 2$ the displacements of the particles are

$$x_A = \tfrac{1}{2}\cos 600 - \tfrac{1}{2}\cos 1039.24 \simeq -0.1$$

$$x_B = \tfrac{1}{2}\cos 600 + \tfrac{1}{2}\cos 1039.24 \simeq -0.9$$

or even more approximately $x_A = 0$ (metres) and $x_B = -1$ (metres).

2. We know two of the eigenvalues: $\lambda = 0$ (since this is a degenerate model) and $\lambda = -2$. So, by inspection, we can write the characteristic equation in the form

$$\lambda(\lambda + 2)(\lambda^2 + 4\lambda + 2) = 0.$$

The other two eigenvalues are therefore obtained by solving the quadratic equation $\lambda^2 + 4\lambda + 2 = 0$. We obtain

$$\lambda = \frac{-4 \pm \sqrt{16 - 8}}{2} = -0.59 \text{ or } -3.41.$$

So the eigenvalues are:

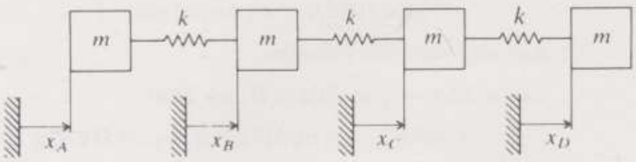
$$\lambda_1 = 0, \quad \lambda_2 = -0.59, \quad \lambda_3 = -2, \quad \lambda_4 = -3.41.$$

The normal mode angular frequencies are given by $\omega = \sqrt{\frac{-\lambda k}{m}}$. That is

$$\omega_1 = 0,\quad \omega_2 \simeq 0.77\sqrt{\frac{k}{m}},\quad \omega_3 \simeq 1.41\sqrt{\frac{k}{m}},\quad \omega_4 \simeq 1.85\sqrt{\frac{k}{m}}$$

(all in rad s^{-1}).

To find the displacement ratios, we write down the equations of motion. The figure below shows the notation I shall use.



The equations of motion are

$$k(x_B - x_A) = m\ddot{x}_A$$
$$-k(x_B - x_A) + k(x_C - x_B) = m\ddot{x}_B$$
$$k(x_D - x_C) - k(x_C - x_B) = m\ddot{x}_C$$
$$-k(x_D - x_C) = m\ddot{x}_D.$$

Substituting $\ddot{x}_A = -\omega^2 x_A$, $\ddot{x}_B = -\omega^2 x_B$, $\ddot{x}_C = -\omega^2 x_C$ and $\ddot{x}_D = -\omega^2 x_D$ we obtain the normal mode eigenvector equations

$$(x_B - x_A) = -\frac{m\omega^2}{k}x_A$$
$$-(x_B - x_A) + (x_C - x_B) = -\frac{m\omega^2}{k}x_B$$
$$(x_D - x_C) - (x_C - x_B) = -\frac{m\omega^2}{k}x_C$$
$$-(x_D - x_C) = -\frac{m\omega^2}{k}x_D.$$

Substituting the highest value of ω into these equations we have:

from the first equation

$$x_B - x_A = -3.41x_A,$$

so $\quad x_B = -2.41x_A$;

from the second equation

$$-x_B\left(1 + \frac{1}{2.41}\right) + x_C - x_B = -3.41x_B,$$

i.e. $\quad -1.41x_B - x_B + x_C = -3.41x_B$,

so $\quad x_C = -x_B$,
$\quad\quad = 2.41x_A$;

from the fourth equation

$$-x_D + x_C = -3.41x_D,$$

i.e. $\quad x_C = -2.41x_D$,

so $\quad x_D = -0.41x_C$
$\quad\quad = (-0.41 \times 2.41)x_A$
$\quad\quad = -x_A$.

Hence putting $x_A = 1$ we obtain the eigenvector $[1\ \ -2.41\ \ 2.41\ \ -1]^T$ so the displacement ratios corresponding to the highest normal mode angular frequency ω_4 are $R_{4B} = -2.41$, $R_{4C} = 2.41$ and $R_{4D} = -1$.

3. (i) From the discussion on vibration absorbers in Section 3 we know that the main mass will remain at rest when the forcing angular frequency is $\sqrt{\frac{k_B}{m_B}}$, where k_B is the stiffness of the absorber springs and m_B is the total absorber mass.

Hence the angular frequency $= \sqrt{\frac{5.9 \times 10^6}{167}} \simeq 188\,\text{rad s}^{-1}$.

When the speed of the shaft is such that the force has this angular frequency the main mass will remain stationary.

(ii) When the main mass is stationary the total force on it must be zero.

The gravitational forces on the main mass and on the absorber masses are cancelled out by the initial compression of the main springs; to put it another way, the gravitational forces are accounted for in determining the equilibrium position from which the vibration displacements are measured. It follows that when the machine is running, the force due to shaft unbalance must be cancelled out by the action of the vibration absorber. So the total maximum force which the absorber exerts on the main mass must be equal and opposite to the force that the unbalanced shaft exerts on the main mass.

Now the amplitude of motion of each absorber mass is 0.001 m — the maximum deflection of absorber spring from the equilibrium position. The stiffness of each absorber spring is $2.95 \times 10^6\,\text{N m}^{-1}$. It follows that the maximum force on the main mass due to each absorber spring is

$$2.95 \times 10^6 \times 0.001 = 2.95 \times 10^3\,\text{N}.$$

So the total maximum force on the main mass due to the absorber is

$$5.9 \times 10^3\,\text{N}.$$

And this must be equal in magnitude (though opposite in direction) to the force on the main mass due to the unbalance of the shaft.

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